

The numbers $L(m, n)$ and
their Relations with Prepared
Bernoulli and Eulerian no

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A formula for the summation of the series s^r has been given by Hansraj Gupta [2] in terms of G-functions. The present author [1] obtained the sum in terms of functions allied to the G-functions. In this paper the sum is obtained in terms of another class of functions termed L-functions. The numerical values of these are symmetrical integrals. These numbers are interesting because of their close relations with Prepared Bernoullian and Eulerian numbers. From the properties of these L-functions it has been possible to express a Prepared Bernoullian number S_{2m+1} in terms of algebraic functions of m although the formula is of no practical utility in the evaluation of S_{2m+1} . Some algebraic identities have also been obtained from the solutions of difference equations connected with L-and related functions.

1. $L(m, n)$ and the series $\sum_{s=1}^n s^r$.

It may be shown that for all positive integers

$$s^r = \sum_{k=1}^r L(r, k) \binom{s+k-1}{r} \quad (1.1)$$

$$\therefore \sum_{s=1}^n s^r = \sum_{k=1}^r L(r, k) \binom{n+k}{r+1} \quad (1.2)$$

$$\text{where } L(m, n) = (m-n+1) L(m-1, n-1) + n L(m-1, n) \} \quad (1.3)$$

$$L(m, 1) = 1, m \geq 1; L(1, n) = 0; n \geq 2$$

Table (1) gives a few values of $L(m, n)$.

2. From (1.3) the first three of the following four can be easily obtained.

$$L(m, n) = L(m, m-n+1) \quad (2.1)$$

$$\sum_{r=1}^m L(m, r) = m! \quad (2.2)$$

$$\sum_{r=1}^{2m} (-1)^{r+1} L(2m, r) = 0 \quad (2.3)$$

$$\sum_{r=1}^{2m+1} (-1)^{r+1} L(2m+1, r) = (-1)^m S_{2m+1} \quad (2.4)$$

where S_{2m+1} are Prepared Bernoullian numbers.

5. Define another set of numbers by the equation

$$L'(m, n) = \sum_{r=1}^{m-n+1} \binom{m-1}{r-1} L(m-r, n-1). \quad (3.1)$$

Table (2) gives a few values of $L'(m, n)$.

It may be easily shown that

$$L'(m, n) = L'(m, m-n+1). \quad (3.2)$$

Further let

$$h(m) = \sum_{r=1}^m (-1)^{r+1} L(m, r) \quad (3.3)$$

$$H(m) = \sum_{r=1}^m (-1)^{r+1} L'(m, r). \quad (3.4)$$

Using (3.1) and (3.3) in (3.4), we get

$$H(m) = 1 - \sum_{s=0}^{m-2} \binom{m-1}{s} h(m-s-1). \quad (3.5)$$

By virtue of (2.1) and (3.2), $h(2r)$ and $H(2r)$ vanish. Hence

$$-h(2n-1) = \sum_{s=1}^{n-1} \binom{2n-1}{2s} h(2n-2s-1). \quad (3.6)$$

Now (3.6) is clearly the recurrence relation between Prepared Bernoullian Numbers S_{2m+1} taken alternately positive and negative.

Hence

$$h(2m+1) = (-1)^m S_{2m+1}. \quad (3.7)$$

(3.7) gives (2.4).

4. By putting $m=2n+1$ in (3.5), it may be shown that

$$H(2m+1) = (-1)^m S_{2m} \quad (4.1)$$

where S_{2m} are Eulerian numbers.

5. Defining another set of numbers by the equation

$$L''(m, n) = \sum_{r=1}^{m-n+1} \binom{m}{r} L(m-r, n-1) \quad (5.1)$$

it may be shown that

$$\sum_{r=1}^m (-1)^r L''(m, r) = \left(\sin \frac{\pi m}{2} + \cos \frac{\pi m}{2} \right) S_m \quad (5.2)$$

where S_{2m+r} are Prepared Bernoullian and S_{2m} are Eulerian numbers.

Table (3) gives a few values of $L''(m, n)$.

6. A solution of the equation (1.3) can be easily obtained by putting $s=1, 2, 3$ etc. in (1.1). We get

$$L(m, n) = \sum_{r=0}^{n-1} (-1)^r \binom{m+1}{r} (n-r)^m. \quad (6.1)$$

7. Using (6.1) as basis let a class of numbers be defined by

$$L(m, n, p) = \sum_{r=0}^{n-1} (-1)^r \binom{m+p}{r} (n-r)^m. \quad (7.1)$$

The following properties may be easily proved.

$$L(m, n, 1) = L(m, n) \quad (7.2)$$

$$L(m, n, p) = (m-n+p) L(m-1, n-1, p) + n L(m-1, n, p) \quad (7.3)$$

$$L(m, n, p) = L(m, n, p-1) - L(m, n-1, p-1) \quad (7.4)$$

$$L(m, n, p) = (-1)^p L(m, m-n+p, p). \quad (7.5)$$

Table (4) gives a few values of $L(m, n, p)$.

8. It is possible to express S_{2m+1} as a finite series in powers of m . However, for the purpose of evaluation of S_{2m+1} , the series is of no use as the expression is not very simple.

From (7.4) we get

$$\sum_{r=1}^{m+p-1} (-1)^r L(m, r, p) = 2 \sum_{r=1}^{m+p-2} (-1)^{r-1} L(m, r, p-1) \quad (8.1)$$

$$= 2^{p-1} \sum_{r=1}^{m+p-2} (-1)^{r-1} L(m, r, 1). \quad (8.2)$$

Using (7.1) and (2.4) in (8.2) we get

$$(-1)^m 2^{p-1} S_{2m+1} = \sum_{t=1}^{2m+1} (-1)^{t+1} t^{2m+1} \sum_{r=0}^{2m+p-t} \binom{2m+p+1}{r}. \quad (8.3)$$

In (8.3) putting $p=1$ we get

$$(-1)^m S_{2m+1} = \sum_{t=1}^{2m+1} (-1)^{t+1} t^{2m+1} \left\{ \sum_{r=0}^{2m-1-t} \binom{2m+2}{r} \right\}. \quad (8.4)$$

9. Multiplying (8.4) by 2^{p-1} and equating to (8.3), we get

$$\begin{aligned} 2^{p-1} \sum_{t=1}^{2m+1} (-1)^{t+1} t^{2m+1} \sum_{r=0}^{2m+1-t} \binom{2m+2}{r} \\ = \sum_{t=1}^{2m+1} (-1)^{t+1} t^{2m+1} \sum_{r=0}^{2m+p-t} \binom{2m+p+1}{r}. \end{aligned} \quad (9.1)$$

From (7.4) and (7.5) it may be shown that $L(2m, m+n, 2n)=0$.

Applying (7.1) to (9.2) we get

$$\sum_{s=0}^{m+n-1} (-1)^s \binom{2m+2n}{s} (m+n-s)^{2m} = 0 \quad (9.3)$$

where $m \geq 1$; $n \geq 1$.

From (7.1) we get

$$L(1, 1, 1) = 1. \quad (9.4)$$

Applying (7.4) repeatedly to itself, we get

$$L(1, n, p) = \sum_{s=0}^k (-1)^s \binom{k}{s} L(1, n-s, p-k). \quad (9.5)$$

Then by using (9.4) in (9.5), we get $L(1, n, p) = (-1)^{n-1} \binom{p-1}{n-1}$. (9.6)

From (7.1) and (9.6), we get

$$\sum_{s=0}^{n-1} (-1)^s (n-s) \binom{p+1}{s} = (-1)^{n-1} \binom{p-1}{n-1} \quad (9.7)$$

where $p \geq n \geq 1$.

References

1. Daljit Singh (1945). "On the series $\sum s^n$." *Math. Student*. Vol. XIII, 2, 59-60.
2. Hansraj Gupta (1940) *Symmetric Functions in the Theory of Integral Numbers*. Lucknow University.

Table (1)

$L(m, n) = (m-n+1) L(m-1, n-1) + n L(m-1, n)$; $L(1, 1) = 1$; $L(1, 2) = 0$; $m, n \geq 1$

$n \backslash m$	1	2	3	4	5	6	7	$\sum_{n=1}^m L(m, n)$	$\sum_{n=1}^m (-1)^{n-1} L(m, n)$
1	1							$1 = 1!$	$+ \quad 1 = +S_1$
2	1	1						$2 = 2!$	0
3	1	4	1					$6 = 3!$	$- \quad 2 = -S_2$
4	1	11	11	1				$24 = 4!$	0
5	1	26	66	26	1			$120 = 5!$	$+ \quad 16 = +S_3$
6	1	57	302	302	57	1		$720 = 6!$	0
7	1	120	1191	2416	1191	120	1	$5040 = 7!$	$- \quad 272 = -S_4$

Table (2)

$$L'(m, n) = \sum_{r=1}^{m-n+1} \binom{m-1}{r-1} L(m-r, n-1)$$

$n \backslash m$	1	2	3	4	5	6	7	$\sum_{n=1}^m L'(m, n)$	$\sum_{n=1}^m (-1)^{n-1} L'(m, n)$
1	1							1	$+1 = +S_0$
2	1	1						2	0
3	1	3	1					5	$-1 = -S_2$
4	1	7	7	1				16	0
5	1	15	33	15	1			65	$+5 = +S_4$
6	1	31	131	131	31	1		326	0
7	1	63	473	883	473	63	1	1957	$-61 = -S_6$

Table (3)

$$L''(m, n) = \sum_{r=1}^{m-n+1} \binom{m}{r} L(m-r, n-1)$$

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$\begin{smallmatrix} n \\ m \end{smallmatrix}$	1	2	3	4	5	6	7	$\sum_{n=1}^m L''(m, n)$	$\sum (-1)^{n-1} L''(m, n)$
1	1							1	+1 = +S ₁
2	1	2						3	-1 = -S ₂
3	1	6	3					10	-2 = -S ₃
4	1	14	22	4				41	+5 = +S ₄
5	1	30	105	65	5			206	+16 = +S ₅
6	1	62	416	581	171	6		1237	-61 = -S ₆
7	1	126	1491	3920	2695	420	7	8560	-272 = -S ₇

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Table (4)

$$L(m, n, p) = (m-n+p) L(m-1, n-1, p) + n L(m-1, n, p)$$

$p=3$							$p=4$						
$\begin{smallmatrix} n \\ m \end{smallmatrix}$	1	2	3	4	5		$\begin{smallmatrix} n \\ m \end{smallmatrix}$	1	2	3	4	5	6
1	1						1	1	-1				
2	1	1					2	1	0	-1			
3	1	4	1				3	1	3	-3	-1		
4	1	11	11	1			4	1	10	0	-10	-1	
5	1	26	66	26	1		5	1	25	40	-40	-25	-1

$\begin{smallmatrix} n \\ m \end{smallmatrix}$	1	2	3	4	5	6	7	$\begin{smallmatrix} n \\ m \end{smallmatrix}$	1	2	3	4	5	6	7	8
1	1	-2	1					1	1	-3	3	-1				
2	1	-1	-1	1				2	1	-2	0	2	-1			
3	1	2	-6	2	1			3	1	1	-8	8	-1	-1		
4	1	9	-10	-10	9	1		4	1	8	-19	0	19	-8	-1	
5	1	24	15	-80	15	24	1	5	1	23	-9	-95	95	9	-23	-1

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The numbers S_n occur in the expansion of $(1-x)^{-1}$ while S_{2m+1} are known.

$$I. S_{2m+1} \equiv (-1)^m$$

The identity

$$S_n = \binom{n}{2} S_{n-2} -$$

gives

$$S_{2m+2} \equiv (-1)^m m$$

Applying (1.2)

$$S_{2m+4} = (-1)^m \cdot 5$$

Again applying

$$S_{2m+6} = (-1)^m \cdot 61$$

and so on.

The above suggests

$$S_{2m+2k} \equiv (-1)^m S_{2k}$$

The following

$$(2m +$$

In (1.1) putting

$$S_{2m+2k+2} = \binom{2m+2k+2}{2}$$

$$+ (-1)^{m-2} \binom{2m+2k+2}{2}$$

$$S_n \equiv (-1)^m \left\{ \binom{2m+2k+2}{2} \right.$$

$$+ \binom{2m+2k+2}{2m+2}$$

$$\equiv (-1)^m \left[\left\{ \binom{2m+2k+2}{2} \right. \right.$$

$$\left. \left\{ \binom{2m+2k+2}{2m+2} \right\} \right]$$