A3226

A18247
A55620
A54869
A18248
A259468,9

# Automorphic Numbers

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An automorphic number is distinguished by having its square end with the number itself. To put this more formally, a number is called automorphic in the scale of notation with base $B$ if all its powers end in the same digits in this scale of notation. If $n$ is the number of digits appearing at the end of every power of the automorphic number $N$, we have $N^x \equiv N(\text{modulo } B^n)$, for $x > 0$. If we let $x = 2$ we have $N^2 \equiv N(\text{modulo } B^n)$, which is sufficient to satisfy the formal definition given above.

In any scale of notation, base $B$, there are two trivial solutions ($N = 0$ and $N = 1$). In base 10, the remaining solutions end in 5 or 6.

If the automorphic numbers of $n$ digits are known, the automorphic numbers of a lesser number of digits can be obtained by dropping the first digit, the first and second digits, etc. Thus, since 625 is automorphic, so also is 25 and 5. It is more difficult, however, to determine the automorphic numbers of $n + 1$, $n + 2, \cdots, n'$ digits, although this was done by one of the authors of this article for $n' = 100$ using nothing more complicated than a desk adding machine. These were first published in *The Fibonacci Quarterly,* Vol. 2, No. 3, page 230 (October 1964).

It is the purpose of this paper to state how the automorphic numbers can be expanded and to describe a practical method of doing this either with simple tools or with a computer.

If one of the automorphic numbers, $N$, of $n$ digits is squared, the last $n$ digits must (by definition) be the same as $N$. If $N$ ends in 5, the next digit to the left is the first digit of $N + 1$; if $N$ ends in 6, the next digit to the left, when subtracted from 10, will produce the first digit of $N + 1$. For example, $5^2 = 25$ so 25 is automorphic; $6^2 = 36$ so 76 is automorphic $(7 = 10 - 3)$. Continuing, $25^2 = 625$ so 625 is automorphic; $76^2 = 5776$ so 376 is automorphic.

If the two automorphic numbers are written, one below the other, it will be seen that the sum of the digits in each column is 9 (in base 10) except for the units where it is 11. Therefore, it is only necessary to develop one of the numbers, and the number ending in 5 is somewhat simpler to work with.

One innovation which was found to be useful (both for computer and "hand" calculation) was to omit all working digits except those necessary to determine the next digit of the automorphic number. Take 2890625, for example. The normal method of squaring is as follows:

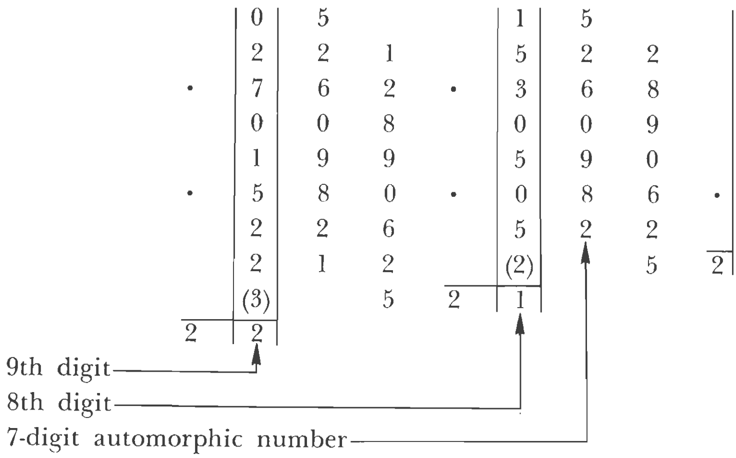
| | |
|---|---|
| | 2890625 |
| | 2890625 |
| | *1*4453125 |
| | *5*781250 |
| | 17*3*43750 |
| | 2601*5*6250 |
| | 23125*0*00 |
| | 57812*5*0 |
| (carry) | 11–12*2*111- - - |
| | 8355712890625 |
| 13-digit automorphic | 9918212890625 |

The only digits in the work which are of any value are the ones in *italics*. The left-hand digits are useless—the right-hand digits are only needed to obtain the "carry." As long as the "carry" can be determined from the previous calculations, the right-hand digits can be omitted.

A method of eliminating all but the necessary digits is shown below. It is based on the Trachtenberg Speed System. Write the largest known automorphic number (ending in 5) vertically with the units digit at the top. In the column immediately to the right, write the same number with the units digit at the bottom, but one line lower on the page than the other column. In the next column to the right indicate with a dot (•) if there is a "carry" from the previous work (this will never exceed 1). Then multiply in accordance with the following rule:

| *First Column* | *Second Column* | *"Dot" Column* |
|---|---|---|
| $A$ | $U$ | • |
| | $T$ | |

Multiply the digit $A$ by the digit $U$ and jot down the *units* digit of the product. Multiply the digit $A$ by the digit $T$ and jot down the *tens* digit of the product. Add these two digits and increase the sum by 1 if there is a dot ("carry") on the same line as $A$. Write this sum on the same line as $A$ in the column to the *left* of the first column. If the sum exceeds 9, write the units digit and a "dot" to indicate a "carry" for determining the next but one digit. After multiplying all the digits of the first column ($A$) by the appropriate digits of the second column ($U$ and $T$), put at the bottom of the column the "column carry" from the previous calculation and then add. The units digit of the sum is the next digit of the automorphic number and the remaining digits are the "column carry." However, if the units digit is odd, the "column carry" must be increased by 1.




When using a computer, the normal method of storing results cannot be employed because of the necessity of developing the product column-by-column rather than line-by-line; which, in turn, requires multiplying individual digits of a number, rather than the number as a whole. However, an IBM 360/30 (64K) computer was able to develop and check 500 digits. The checking was done after the required number of digits of the "5" number had been obtained. The corresponding "6" number was determined as explained previously and squared. However, it was squared column-by-column, and after each column was added, the units digit was compared with corresponding digit of the number, before proceeding with the next column.

As mentioned, these numbers were first expanded to 100 digits by "hand." With a desk adding machine available, it was found most satisfactory to develop six digits at a time. This involved some special procedures which would require too much space to describe in an article of this length. After each six digits had been determined the results were verified by using the number ending in 6.

The non-trivial 1000-digit automorphic numbers in bases 6, 10, and 12 are printed out in full, after the following commentary by the editor of this journal.

*Editor's Comments:* Almost simultaneously with the receipt of the manuscript of this article, the editor received a number of computer-calculated automorphic numbers. The authors of this article had submitted 500-digit pairs in bases 6 and 10. Then M. F. Jones, of the Computer Automation Systems, Ltd., in London, sent a pair of base-10 1500-digit automorphs. Larry Dean Morse of Rolla, Missouri sent in 1000-digit pairs in bases 6 and 12, and 2000-digit pairs in base 10. The climax (for the present, at least) was reached when the editor received, through the courtesy of Martin Gardner, the monstrous 22,300-digit automorphic number ending in 5. This had been calculated by Jerry Feinberg and Terry Moore of the California Institute of Technology with the help of a computer with a sense of humor: the print-out asks the reader to "Enjoy, Enjoy, Enjoy" and to "Square Me If You Can!"

Automorphic numbers have been studied for many years, but the recent interest has come only after a number of years of apparent lack of interest. Maurice Kraitchik, in his *Mathematical Recreations* (W. W. Norton & Co., Inc., N. Y., 1942) gave the 16-digit pair in base 10. This same pair is quoted in a number of books on mathematical recreations—with no attempt to add even one extra digit. Then, in 1964, J. A. H. Hunter's 17-digit pair was published in *Recreational Mathematics Magazine* (No. 14, Jan-Feb 1964, page 19). R. A. Fairbairn took up the challenge immediately, publishing the 100-digit pair in *The Fibonacci Quarterly.*

More recently, Martin Gardner quoted one of the 100-digit number in his "Mathematical Games" column in the January 1968 issue of *Scientific American* (pages 124-127). That produced the spate of 1000-and-more-digit endeavors.

R. A. Fairbairn points out that the number of automorphic numbers (of any given number of digits, and including the trivial 0 and 1) in any base can be calculated. The number of automorphic numbers in base $b$ is $2^p$ where $p$ is the number of distinctly different primes in the factorization of $b$. For example, $6 = (2)(3)$, so there are $2^2$ or four automorphic numbers in base 6; $12 = (2^2)(3)$, so there are also $2^2$ or four automorphic numbers in base 12. In base 2310, since $2310 = (2)(3)(5)(7)(11)$, there are $2^5$ or 32 automorphic numbers: those ending in 0 and 1, and those pairs ending in the following "digits" (each of the four-digit integers listed is actually the terminal "digit"—in base 2310—of the 15 pairs of automorphic numbers):

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0210 | 0231 | 0330 | 0385 | 0441 | 0540 | 0561 | 0595 |
| 2101 | 2080 | 1981 | 1926 | 1870 | 1771 | 1750 | 1716 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0616 | 0715 | 0771 | 0826 | 0925 | 0946 | 1155 |
| 1695 | 1596 | 1540 | 1485 | 1386 | 1365 | 1156 |

The sum of the terminal digits of the non-trivial automorphic number pairs in any base $b$ is $b + 1$; the sum of the other corresponding digit-pairs in base $b$ is $b - 1$.

Although this editor will always welcome larger and larger automorphic numbers in any base, he cannot anticipate publishing any in JRM after this issue. More extensive evaluations must be published elsewhere—perhaps in a tentatively-planned *Mathematical Tables for Recreational Mathematicians.*

*Vernon deGuerre,* employed by the Borough of North York, Metropolitan Toronto, as computer programmer is presently attending a computer course at York University, Toronto, under the direction of Dr. P. Rajagopal, who was most helpful in suggesting improvements which have been incorporated in this article.

*R. A. Fairbairn* is a retired engineer of The Bell Telephone Company of Canada.

The Non-Trivial Automorphic Numbers, Base 6 to 1000 Digits

53543 30200 25150 34425 10341 01350 23342 41452 33333 31105
34111 55403 15010 23233 43551 23230 22333 40243 11541 22344
21113 52055 41520 15332 44243 43504 32151 53040 22415 32512
24544 45230 42031 52324 52305 20534 54435 01511 54243 41555
41515 42020 10525 30044 32311 25235 15131 13023 30451 30505

14041 01311 20315 51200 10322 20520 23145 10432 14140 50535
23504 01225 10550 10523 02442 35530 22214 44500 21331 02340
51234 24054 22230 51333 44421 24103 14143 53401 41354 42500
30531 01252 22105 21154 03112 15254 12432 34344 02305 15013
35243 53233 53152 22220 14214 34152 03515 42235 52433 30341

50022 31551 34244 50343 42111 20010 23210 13344 33004 30533
15540 22225 42052 05211 53345 05111 15105 54321 31130 51220
32550 55135 43205 45214 23350 23004 15424 41322 22024 34242
00424 25345 30353 25312 41521 02513 23435 25543 13405 54040
40035 14254 21450 34304 10120 53532 54030 33554 22054 31315

10155 33524 23551 41224 53451 24435 24205 45402 43203 41021
41412 10414 04315 20555 21325 13552 31551 52531 31015 25051
22450 12353 30231 00403 35415 52351 24250 22532 55332 35332
33125 13545 02150 45245 31544 51010 42321 00300 12522 35130
34535 14224 51331 52214 20003 05000 34241 40040 33342 05344 A55620

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02012 25355 30405 21130 45214 54205 32213 14103 22222 24450
21444 00152 40545 32322 12004 32325 33222 15312 44014 33211
34442 03500 14035 40223 11312 12051 23404 02515 33140 23043
31011 10325 13524 03231 03250 35021 01120 54044 01312 14000
14040 13535 45030 25511 23244 30320 40424 42532 25104 25050

41514 54244 35240 04355 45233 35035 32410 45123 41415 05020
32051 54330 45005 45032 53113 20025 33341 11055 34224 53215
04321 31501 33325 04222 11134 31452 41412 02154 14201 13055
25024 54303 33450 34401 52443 40301 43123 21211 53250 40542
20312 02322 02403 33335 41341 21403 52040 13320 03122 25214

05533 24004 21311 05212 13444 35545 32345 42211 22551 25022
40015 33330 13503 50344 02210 50444 40450 01234 24425 04335
23005 00420 12350 10341 32205 32551 40131 14233 33531 21313
55131 30210 25202 30243 14034 53042 32120 30012 42150 01515
15520 41301 34105 21251 45435 02023 01525 22001 33501 24240

45400 22031 32004 14331 02104 31120 31350 10153 12352 14534
14143 45141 51240 35000 34230 42003 24004 03024 24540 30504
33105 43202 25324 55152 20140 03204 31305 33023 00223 20223
22430 42010 53405 10310 24011 04545 13234 55255 43033 20425
21020 41331 04224 03341 35552 50555 21314 15515 22213 50213 A54869

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## The Non-Trivial Automorphic Numbers, Base 10 to 1000 Digits

12781 25400 13369 00860 34889 08436 40238 75765 93682 19796
26181 91783 35204 92704 19932 48752 37825 86714 82789 05344
89744 01426 12317 03569 95484 19499 44461 06081 46207 25403
65599 98271 58835 60350 49327 79554 07419 61849 28095 20937
53026 85239 09375 62839 14857 16123 67351 97060 92242 42398

77700 75749 55787 27155 97674 13458 99753 76955 15862 71888
79415 16307 56966 88163 52155 04889 82717 04378 50802 84340
84412 64412 68218 48514 15772 99160 34497 01789 23357 96684
99144 73895 66001 93254 58276 78000 61832 98544 26232 82725
75561 10733 16069 70158 64984 22229 12554 85729 87933 71478

66323 17240 55157 56102 35254 39949 99345 60808 38011 90741
53006 00560 55744 81870 96927 85099 77591 80500 75416 42852
77081 62011 35024 68060 58163 27617 16767 65260 93752 80568
44214 48619 39604 99834 47280 67219 06670 41724 00942 34466
19781 24266 90787 53594 46166 98508 06463 61371 66384 04902

92193 41881 90958 16595 24477 86184 61409 12878 29843 84317
03248 17342 88865 72737 66314 65191 04988 02944 79608 14673
76050 39571 96893 71467 18013 75619 05546 29968 14764 26390
39530 07319 10816 98029 38509 89006 21665 09580 86381 10005
57423 42323 08961 09004 10661 99773 92256 25991 82128 90625 A18247

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87218 74599 86630 99139 65110 91563 59761 24234 06317 80203
73818 08216 64795 07295 80067 51247 62174 13285 17210 94655
10255 98573 87682 96430 04515 80500 55538 93918 53792 74596
34400 01728 41164 39649 50672 20445 92580 38150 71904 79062
46973 14760 90624 37160 85142 83876 32648 02939 07757 57601

22299 24250 44212 72844 02325 86541 00246 23044 84137 28111
20584 83692 43033 11836 47844 95110 17282 95621 49197 15659
15587 35587 31781 51485 84227 00839 65502 98210 76642 03315
00855 26104 33998 06745 41723 21999 38167 01455 73767 17274
24438 89266 83930 29841 35015 77770 87445 14270 12066 28521

33676 82759 44842 43897 64745 60050 00654 39191 61988 09258
46993 99439 44255 18129 03072 14900 22408 19499 24583 57147
22918 37988 64975 31939 41836 72382 83232 34739 06247 19431
55785 51380 60395 00165 52719 32780 93329 58275 99057 65533
80218 75733 09212 46405 53833 01491 93536 38628 33615 95097

07806 58118 09041 83404 75522 13815 38590 87121 70156 15682
96751 82657 11134 27262 33685 34808 95011 97055 20391 85326
23949 60428 03106 28532 81986 24380 94453 70031 85235 73609
60469 92680 89183 01970 61490 10993 78334 90419 13618 89994
42576 57676 91038 90995 89338 00226 07743 74008 17871 09376 A18248

## The Non-Trivial Automorphic Numbers, Base 12 to 1000 Digits

02638 40220 689A9 212B3 31244 84B1B 15866 73751 B4214 B0337
A5B4A 67299 A2409 57340 13578 23BA8 1628B 29304 47917 57518
03B80 9154A 19303 24718 10916 8829B 4067A 32043 221B5 B0294
25086 A5721 58B82 17B61 55475 B8B80 BB82B BA2B0 67419 26623
32056 51B63 2756A 65428 673A0 80213 B3BA1 89B76 0A134 6BAA7

6842B 5893B 5A2A7 44720 13415 BB842 A4138 9314A 24113 64156
7300A 59B5A A31A5 79041 BB6A8 40758 93874 52544 41250 34952
8B825 1025A 33B56 A1538 0A933 98270 29239 03B69 390A1 6BB15
0B709 01208 6308A 3A159 A8744 B6014 9BAB7 56628 75B2B B5564
A0149 46929 61364 A3474 69919 55778 B7642 6A509 89761 27955

AA3B6 82639 B99A6 40525 26169 3B1AA 6A655 9A029 09A78 A7A51
B6475 22A97 87775 26213 7BAA4 91655 B8383 12B16 A3111 5620B
8425A 746B0 B0607 B021B 17B39 B2643 102A8 41559 0B84A 64BAB
72943 763B4 31757 13386 24146 78702 30478 62478 63268 13990
B6A84 48895 23778 00861 94532 99042 12315 55A08 99A24 50318

70A45 0A253 0769A 08682 62B14 00167 78442 30904 79B6B 21001
3736A 3B4B9 843B2 42538 67A44 A8A03 38504 32547 97647 B6B71
B0024 4855A 5960B 39854 AA763 13BA1 A5163 79014 20433 5A1AA
78B11 3A4B2 A1B2B 37062 787B5 02745 66720 B2A15 25678 B7901
35998 30402 B6468 A133A 81A3A 16986 B267B 3452B 21B61 B3854 A259468

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B9583 7B99B 53212 9A908 8A977 370A0 A6355 4846A 079A7 0B884
16071 54922 197B2 6487B A8643 98013 A5930 928B7 742A4 646A3
B803B 2A671 A28B8 974A3 AB2A5 33920 7B541 89B78 99A06 0B927
96B35 1649A 63039 A405A 66746 0303B 00390 0190B 547A2 95598
89B65 6A058 94651 56793 5481B 3B9A8 0801A 32045 B1A87 50114

53790 63280 61914 7749B A87A6 00379 17A83 28A71 97AA8 57A65
48BB1 62061 18A16 42B7A 00513 7B463 28347 69677 7A96B 87269
30396 AB961 88065 1A683 B1288 2394B 92982 B8052 82B1A 500A6
B04B2 BA9B3 58B31 81A62 13477 05BA7 20104 65593 46090 06657
1BA72 75292 5A857 18747 522A2 66443 04579 516B2 3245A 94266

11805 39582 02215 7B696 95A52 80A11 51566 21B92 B2143 1416A
05746 99124 34446 959A8 40117 2A566 03838 A90A5 18AAA 659B0
37961 4750B 0B5B4 0B9A0 A4082 09578 AB913 7A662 B0371 57010
49278 45807 8A464 A8835 97A75 434B9 8B743 59743 58953 A822B
05137 73326 98443 BB35A 27689 22B79 A98A6 661B3 22197 6B8A3

4B176 B1968 B4521 B3539 590A7 BBA54 43779 8B2B7 42050 9ABBA
84851 80702 37809 79683 54177 131B8 836B7 89674 24574 0504A
0BB97 73661 625B0 82367 11458 A801A 16A58 42BA7 9B788 61A11
430AA 81709 1A090 84B59 43406 B9476 5549B 091A6 96543 042BA
86223 8B7B9 05753 1A881 3A181 A5235 09540 87690 9A05A 08369 A259469