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# PROOF OF A CONJECTURE OF RAMANUJAN

by A. O. L. ATKIN

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## 1. Introduction. We write

$$f(x) = (1-x)(1-x^2)(1-x^3) \dots$$

and

$$\sum_{n=0}^{\infty} p(n)x^n = 1/f(x),$$

so that  $p(n)$  is the number of unrestricted partitions of  $n$ . Ramanujan [1] conjectured in 1919 that if  $q = 5, 7$ , or  $11$ , and  $24m \equiv 1 \pmod{q^n}$ , then  $p(m) \equiv 0 \pmod{q^n}$ . He proved his conjecture for  $n = 1$  and  $2$ †, but it was not until 1938 that Watson [4] proved the conjecture for  $q = 5$  and all  $n$ , and a suitably modified form for  $q = 7$  and all  $n$ . (Chowla [5] had previously observed that the conjecture failed for  $q = 7$  and  $n = 3$ .) Watson's method of modular equations, while theoretically available for the case  $q = 11$ , does not seem to be so in practice even with the help of present-day computers. Lehner [6, 7] has developed an essentially different method, which, while not as powerful as Watson's in the cases where  $\Gamma_0(q)$  has genus zero, is applicable in principle to all primes  $q$  without prohibitive calculation. In particular he proved the conjecture for  $q = 11$  and  $n = 3$  in [7]. Here I shall prove the conjecture for  $q = 11$  and all  $n$ , following Lehner's approach rather than Watson's. I also prove the analogous and essentially simpler result for  $c(m)$ , the Fourier coefficient‡ of Klein's modular invariant  $j(\tau)$  as

**THEOREM 1.** If  $m \equiv 0 \pmod{11^n}$ , then  $c(m) \equiv 0 \pmod{11^n}$ .

The full truth with regard to Ramanujan's original conjecture is thus now known to be: If  $24m \equiv 1 \pmod{5^a 7^b 11^c}$ , then  $p(m) \equiv 0 \pmod{5^a 7^b 11^c}$ , where  $\beta = [(b+2)/2]$ .

In view of Watson's result we need only prove here

**THEOREM 2.** If  $24m \equiv 1 \pmod{11^n}$ , then  $p(m) \equiv 0 \pmod{11^n}$ .

The general plan of the paper is as follows. In §2 we describe the notation and general theory required for the proof of Theorem 1. In §3 we carry through sufficient detailed calculation to prove Theorem 1. In §4 the additional theory required for the proof of Theorem 2 is given, and in §5 Theorem 2 is proved. Necessary calculations which would unduly interrupt the main argument are given in Appendices.

## 2. Functions on $\Gamma_0(11)$ .

**2.1.** We consider the subgroup  $\Gamma_0(11)$  of the full modular group  $\Gamma(1)$ , defined by those transformations

$$\tau \rightarrow V\tau = \frac{a\tau + b}{c\tau + d} \quad (a, b, c, d, \text{ integral with } ad - bc = 1)$$

† Ramanujan [1, 2]. See also Rushforth [3].

‡ We take the Fourier series of  $j(\tau)$  with leading coefficient unity and constant term zero. Thus  $j(\tau) = x^{-1} + 196884x + \dots$  with  $x = e^{2\pi i \tau}$ .

of  $\Gamma(1)$  that satisfy  $c \equiv 0 \pmod{11}$ .  $\Gamma_0(11)$  is of genus 1, and its fundamental region has two cusps  $\tau = i\infty$  and  $\tau = 0$ , with local variables  $x = e^{2\pi i\tau}$ ,  $x = e^{-2\pi i/11\tau}$ , respectively. By "an entire modular function on  $\Gamma_0(11)$ " we understand a function  $F(\tau)$ , regular in  $\text{Im } \tau > 0$ , that satisfies  $F(V\tau) = F(\tau)$  for  $V \in \Gamma_0(11)$ , and has at most polar singularities in the local variables at the two cusps of  $\Gamma_0(11)$ . For such  $F(\tau)$  we shall write  $F \in S$ . If in addition  $F(\tau)$  is zero at  $\tau = i\infty$  we write  $F \in S^\infty$ . Finally, if  $F(\tau)$  is zero at  $\tau = 0$  we write  $F \in S^0$ .

We refer to the expansion of  $F(\tau)$  in powers of  $x = e^{2\pi i\tau}$  at  $\tau = i\infty$  as its Fourier series (FS).

We have:

LEMMA 1. If  $F(\tau) \in S$ , then  $F^*(\tau) = F(-1/11\tau) \in S$ .

A simple proof is given by Newman [9, Lemma 1]. It is clear that the expansion of  $F(\tau)$  at  $\tau = 0$  is the FS of  $F^*(\tau)$ , and that  $F \in S^\infty$ ,  $S^0 \Leftrightarrow F^* \in S^0, S^\infty$ .

We now introduce a linear operator  $U$  defined by

$$\left. \begin{aligned} 11UF(\tau) &= \sum_{r=0}^{10} F\left(\frac{\tau+r}{11}\right), \\ U^{n+1}F(\tau) &= U(U^n F(\tau)) \quad (n \geq 1). \end{aligned} \right\} \quad (1)$$

Clearly

$$U(a_1 F_1 + a_2 F_2) = a_1 U F_1 + a_2 U F_2,$$

if  $a_1, a_2$  are constants. If the FS of  $F(\tau)$  is

$$\sum_{r=r_0}^{\infty} \alpha_r x^r,$$

then the FS of  $UF(\tau)$  is

$$\sum_{11r \geq r_0} \alpha_{11r} x^r.$$

By  $UF(-1/11\tau)$  we shall understand the effect of replacing  $\tau$  by  $-1/11\tau$  in  $UF(\tau)$  and not " $UG(\tau)$  where  $G(\tau) = F(-1/11\tau)$ ".

We also write

$$F_1(\tau) \equiv F_2(\tau) \pmod{m} \quad (2)$$

if all the respective coefficients in the FS of  $F_1(\tau)$  and  $F_2(\tau)$  are congruent modulo  $m$ . Thus nothing is asserted by (2) as to the expansions at  $\tau = 0$ .

It will be convenient in the sequel to assess divisibility by powers of 11 by using an exponential valuation. Accordingly, for integral  $a$ , we define  $\pi(a)$  by

$$11^{\pi(a)} \mid a, \quad 11^{\pi(a)+1} \nmid a,$$

and for rational  $a = b/c$  we define

$$\pi(a) = \pi(b) - \pi(c).$$

We write conventionally  $\pi(0) = \infty$ , and regard any inequality  $\pi(0) \geq k$  as valid.

We have

$$\begin{aligned}\pi(ab) &= \pi(a) + \pi(b), \\ \pi(a+b) &\geq \min(\pi(a), \pi(b)),\end{aligned}$$

with equality if  $\pi(a) \neq \pi(b)$ .

The crucial results on  $UF(\tau)$  are given by Lehner (Theorem 8 and (8.81) of [6]), and are as follows.

LEMMA 2. If  $F(\tau) \in S$ , then

$$(i) \quad UF(\tau) \in S, \quad (ii) \quad 11UF(-1/11\tau) - 11UF(11\tau) = F(-1/121\tau) - F(\tau).$$

Note that in (ii)  $UF(11\tau)$  and  $F(-1/121\tau)$  are not themselves in  $S$ . It is also immediate that

$$F(\tau) \in S^\infty \Rightarrow UF(\tau) \in S^\infty.$$

((4) is not valid for  $S_0$ .)

Reverting now to the proof of Theorem 1, we see that, since  $j(\tau) \in S$ , then  $U^n j(\tau) \in S^\infty$  for  $n \geq 1$ . Theorem 1 is then equivalent to proving that the FS of  $11^{-n} U^n j(\tau)$  has integral coefficients. To establish this, we obtain first a standard basis for the functions of  $S^\infty$ , and then use Lemma 2 to obtain detailed information as to the effect of the operator  $U$  on these functions.

2.2. A linear basis for functions on  $\Gamma_0(11)$ . The following lemma is proved in Appendix A.

LEMMA 3. For all integral  $n \geq 2$ , there exist functions  $G_n(\tau)$ ,  $g_n(\tau)$ ,  $h_n(\tau)$  with the following properties:

$$\begin{aligned}(i) \quad & G_n(\tau) \in S^0, \quad g_n(\tau) \in S^\infty, \quad h_n(\tau) \in S^\infty, \\ (ii) \quad & G_n(-1/11\tau) = h_n(\tau) = 11^{\theta(n)} g_n(\tau),\end{aligned}$$

where

$$\theta(n) = 6k + 2, 3, 4, 6, 6$$

according as

$$n = 5k + 2, 3, 4, 5, 6 \quad (k \geq 0).$$

(iii) The FS of  $G_n(\tau)$  has integral coefficients with leading term  $x^{-n}$ .

(iv) The FS of  $g_n(\tau)$  has integral coefficients with leading term  $x^{\psi(n)}$ ,

where

$$\psi(n) = 5k + 1, 2, 3, 5, 4$$

according as

$$n = 5k + 2, 3, 4, 5, 6 \quad (k \geq 0).$$

Further, there exists a function  $B(\tau) \in S$  with simple poles at  $\tau = 0$  and  $\tau = i\infty$ , such that  $B(-1/11\tau) = B(\tau)$ . The FS of  $B(\tau)$  has integral coefficients, with leading term  $x^{-1}$ .

Since the Riemann surface of  $\Gamma_0(11)$  cannot support a univalent function, we have the immediate corollary:

LEMMA 4. Suppose that  $F(\tau) \in S$  has a pole of order  $M$  at  $\tau = 0$  and a pole of order  $N$  at  $\tau = i\infty$ . Then

$$F(\tau) = \sum_{r=2}^N \lambda_{-r} G_r(\tau) + \lambda_{-1} B(\tau) + \lambda_0 + \sum_{r=2}^M \lambda_r h_r(\tau),$$

where the  $\lambda_r$  ( $-N \leq r \leq M$ ) are constants.

Finally we restate Lemma 4 in the case of greatest interest to us.

LEMMA 5. Suppose that  $F(\tau) \in S^\infty$  has a pole of order  $M$  at  $\tau = 0$ . Then

$$F(\tau) = \sum_{r=2}^M \lambda_r h_r(\tau), \quad F(-1/11\tau) = \sum_{r=2}^M \lambda_r G_r(\tau).$$

For a given  $F(\tau) \in S^\infty$ , the constants  $\lambda_r$  in Lemma 5 can be determined from the FS of either  $F(-1/11\tau)$  or  $F(\tau)$ . We are mainly concerned not with the exact value of  $\lambda_r$ , but with  $\pi(\lambda_r)$ . In §3 below we obtain suitable lower bounds for  $\pi(\lambda_r)$  in the case when  $F(\tau) = U g_n(\tau)$ . The calculations take a simpler form when we consider  $F(\tau) = U h_n(\tau)$ ; the transition to  $U g_n(\tau)$  is immediate from Lemma 3(ii).

3.1. Since  $h_n(\tau) \in S^\infty$ , we have

$$11U h_n(\tau) = \sum c_{nr} h_r(\tau), \quad (5)$$

where the  $c_{nr}$  are constants, by Lemma 5. It is convenient to regard the sum in (5) as one from  $r = 2$  to  $\infty$ , although all but a finite number of the  $c_{nr}$  are zero. We have also

$$11U h_n(-1/11\tau) = \sum c_{nr} G_r(\tau) \quad (6)$$

and, by Lemma 2(ii),

$$11U h_n(-1/11\tau) - 11U h_n(11\tau) = -h_n(\tau) + h_n(-1/121\tau). \quad (7)$$

It follows from (7) that the principal part of the FS of  $11U h_n(-1/11\tau)$  is the same as that of  $h_n(-1/121\tau) = G_n(11\tau)$ , since  $U h_n(11\tau)$  and  $h_n(\tau)$  are zero at  $\tau = i\infty$ . Hence the coefficients  $c_{nr}$  may be uniquely determined by the fact that the FS of

$$G_n(11\tau) - \sum c_{nr} G_r(\tau)$$

has no terms in  $x^{-11n}, \dots, x^{-3}, x^{-2}$ . It will then necessarily have no term in  $x^{-1}$ , which provides a check in numerical work. It follows that the  $c_{nr}$  are integers (since each  $G_r$  has leading term  $x^{-r}$ , and the FS of  $G_n(11\tau)$  has integral coefficients) and also that

$$c_{nr} = 0 \quad \text{if } r > 11n. \quad (8)$$

Considering next the determination of the  $c_{nr}$  from (5) we observe that for different  $r$  the FS of  $h_r(\tau)$  commence with different powers† of  $x$ , by Lemma 3(iv).

† The linear basis used by Lehner [6, 7] does not have this property.



Thus, since every coefficient in the FS of  $11Uh_n(\tau)$  is divisible by  $11^{\theta(n)+1}$ , and the leading term in the FS of  $h_r(\tau)$  is  $11^{\theta(r)}x^{\psi(r)}$ , we have

$$\pi(c_{nr}) \geq \theta(n) - \theta(r) + 1 \quad (9)$$

and

$$c_{nr} = 0 \quad \text{if} \quad 11\psi(r) < \psi(n). \quad (10)$$

We now establish certain conditions under which  $\pi(c_{nr}) \geq 3$ .

**3.2. The values of  $c_{nr} \pmod{11^3}$ .** All congruences in this section are to the modulus  $11^3$ . It follows from (6) and (7) that

$$G_2(11\tau) - 11^2 g_2(\tau) \equiv \sum c_{2r} G_r(\tau), \quad G_n(11\tau) \equiv \sum c_{nr} G_r(\tau) \quad (n \geq 3). \quad (11)$$

We shall use the symbol  $(k, l, m)$  to denote an expression of the form

$$11^2 \sum_{i=k}^{l-1} \lambda_i G_i(\tau) + 11 \sum_{i=l}^{m-1} \lambda_i G_i(\tau) + \sum_{i=m}^N \lambda_i G_i(\tau),$$

where  $k < l < m \leq N$  and the  $\lambda_i$  are integral constants. Then by direct calculation† we find

$$G_2(11\tau) - 11^2 g_2(\tau) \equiv (2, 9, 19), \quad G_3(11\tau) \equiv (8, 18, 28). \quad (12)$$

Now Table 5 in Appendix B shows that

$$G_i(\tau) G_j(\tau) = \sum_{r=-3}^0 \mu_r G_{i+j+r}(\tau),$$

where the  $\mu_r$  are integral, and  $\mu_{-3}, \mu_{-2} \equiv 0 \pmod{11}$ , for all  $i$  and  $j$ . It follows that

$$\begin{aligned} (k_1, l_1, m_1)(k_2, l_2, m_2) &\equiv (k_3, l_3, m_3), \\ \text{where} \quad k_3 &= \min(k_1 + m_2 - 1, k_2 + m_1 - 1, l_1 + l_2 - 1), \\ l_3 &= \min(l_1 + m_2 - 1, l_2 + m_1 - 1), \\ m_3 &= m_1 + m_2 - 1. \end{aligned}$$

Further, from Table 6 we have, for  $m \geq 4$ ,

$$11^2 g_2(\tau). (k, l, m) \equiv (m-2, \infty, \infty).$$

Thus

$$\left. \begin{aligned} G_4(11\tau) &= G_2^2(11\tau) - 11G_3(11\tau) \equiv (17, 27, 37), \\ G_5(11\tau) &= G_2(11\tau)G_3(11\tau) - 11G_4(11\tau) \equiv (26, 36, 46), \\ G_6(11\tau) &= G_2(11\tau)G_4(11\tau) \equiv (35, 45, 55), \\ G_7(11\tau) &= G_2(11\tau)G_5(11\tau) \equiv (44, 54, 64). \end{aligned} \right\} \quad (13)$$

† This was done in three different ways on three different machines: firstly using Lemma 9 on a Diehl desk calculator at Durham University; next using (5) on an Elliott 803 computer at Durham University; and finally using (6) on the I.C.T. Atlas 1 computer at Chilton. The computing times were respectively one week, one hour, and ten seconds.

It is now easily seen by induction, since  $G_{n+5}(\tau) = G_n(\tau)G_5(\tau)$ , that

$$G_n(11\tau) \equiv (9n-19, 9n-9, 9n+1) \quad (n \geq 3), \quad (14)$$

and thus, by (11),

$$\pi(c_{nr}) \geq 3 \quad \text{if } r \leq 9n-20, n \geq 3. \quad (15)^\dagger$$

**3.3. The values of  $c_{nr} \pmod{11^4}$ .** We have  $11Uh_n(\tau) \equiv 0 \pmod{11^4}$  for  $n \geq 3$  (since  $\theta(n)+1 \geq 4$ ), and (5) with  $\pi(c_{21}) \geq 2$ ,  $\pi(c_{22}) \geq 1$  gives  $11Uh_2(11\tau) \equiv 0 \pmod{11^4}$ .

Then, by (6) and (7),

$$\left. \begin{aligned} G_2(11\tau) - 11^2 g_2(\tau) &\equiv \sum c_{2r} G_r(\tau) \pmod{11^4}, \\ G_3(11\tau) - 11^2 g_3(\tau) &\equiv \sum c_{3r} G_r(\tau) \pmod{11^4}, \\ G_n(11\tau) &\equiv \sum c_{nr} G_r(\tau) \pmod{11^4} \quad (n \geq 4). \end{aligned} \right\} \quad (16)$$

Hence, by arguments similar to those of §3.2, we obtain

$$c_{5r} \equiv 0 \pmod{11^4} \text{ for } r \leq 15, \quad c_{7r} \equiv 0 \pmod{11^4} \text{ for } r \leq 33.$$

A crude induction, using Table 5 and (15), now shows that

$$c_{5k+2,r} \equiv 0 \pmod{11^4} \text{ for } r \leq 15k+18, \quad k \geq 1. \quad (17)$$

We summarise our results on  $c_{nr}$  in the forms actually required later.

#### LEMMA 6.

|                                                      |                                                                                                               |                |
|------------------------------------------------------|---------------------------------------------------------------------------------------------------------------|----------------|
| $\pi(c_{nr}) \geq 0$                                 | <i>always,</i>                                                                                                | (from §3.1)    |
| $\pi(c_{nr}) \geq \theta(n) - \theta(r) + 1$         | <i>always,</i>                                                                                                | (from (9))     |
| $\pi(c_{nr}) \geq 1$                                 | <i>for <math>n = 2</math> or <math>3</math>, <math>9 \leq r \leq 11</math>,</i>                               | (from (12))    |
| $\pi(c_{nr}) \geq 2$                                 | <i>for <math>n = 2</math> or <math>3</math>, <math>r \leq 8</math>,</i>                                       | (from (12))    |
| $\pi(c_{nr}) \geq 3$                                 | <i>for <math>n = 4</math>, <math>r \leq 16</math>,</i>                                                        | (from (15))    |
| $\pi(c_{nr}) \geq 3$                                 | <i>for <math>n \geq 5</math>, <math>r \leq n+14</math>,</i>                                                   | (from (15))    |
| $\pi(c_{nr}) \geq 4$                                 | <i>for <math>n \equiv 2 \pmod{5}</math>, <math>n \geq 7</math>, <math>r = n-1</math> or <math>n-2</math>,</i> | (from (17))    |
| $\pi(c_{22}) = 2, \pi(c_{32}) = 3, \pi(c_{42}) = 4.$ |                                                                                                               | (from Table 7) |

**3.4.** We now use the results of §3.3 to show that, in effect, functions of a suitable form remain of that form under the operation  $11^{-1}U$ . This is the basis of the proofs of Theorems

<sup>†</sup> This result is by no means best possible. We can, by consideration of cases  $\pmod{5}$ , establish results with  $11n$  instead of  $9n$  on the right-hand side of (14), but (15) suffices later.

### 3.5. Proof of Theorem 1.

We may express Lemma 2 (ii), in the form:

LEMMA 9. *If  $F(\tau) \in S$ , then  $F(-1/11\tau) + 11UF(\tau)$  is an entire function on the full modular group  $\Gamma(1)$ .*

Choosing  $F(\tau) = B(\tau)$ , we have, since the FS of  $B(\tau)$  is  $x^{-1} - 5 + \dots$ ,

$$60 + B(\tau) + 11UB(\tau) = j(\tau).$$

Now  $U\{B(\tau) + 5\} \in S^\infty$ , and  $11UB(-1/11\tau)$  has FS  $x^{-11} + O(x^{-1})$ . Hence

$$11\{UB(-1/11\tau) + 5\} = \sum_{n=2}^{11} \alpha_n G_n(\tau),$$

where the  $\alpha_n$  are integral constants, and so

$$11^{-1}\{UB(\tau) + 5\} = \sum_{n=2}^{11} \alpha_n 11^{\theta(n)-2} g_n(\tau) \in X.$$

Thus

$$11^{-1}Uj(\tau) = 11^{-1}U\{B(\tau) + 5\} + U^2\{B(\tau) + 5\} \in X,$$

by (28). Repeated application of (28) shows that  $11^{-n}U^n j(\tau) \in X$  for  $n \geq 1$ .

Now the FS of any function in  $X$  has integral coefficients, while the FS of  $U^n j(\tau)$  is  $\sum_{m=1}^{\infty} c(11^n m) x^m$ . Hence for all  $m \geq 1$ ,  $n \geq 1$  we have that  $11^{-n}c(11^n m)$  is an integer, which is

Theorem 1.

Theorem 1 is best possible in the sense that  $c(11^n) \not\equiv 0 \pmod{11^{n+1}}$ . We have

$$11^{-1}Uj(\tau) \equiv \alpha_2 g_2(\tau) \pmod{11}.$$

Now  $\alpha_2 = 1627$  and so  $\pi(\alpha_2) = 0$ . Hence, by repeated application of Lemma 8, we have (since  $Y^0 \subseteq X^0$ )

$$11^{-n}U^n j(\tau) \in X^0,$$

and so

$$11^{-n}U^n j(\tau) \equiv k_n g_2(\tau) \pmod{11},$$

where  $\pi(k_n) = 0$ .

Thus

$$11^{-n}c(11^n) \equiv k_n \pmod{11},$$

and so

$$c(11^n) \not\equiv 0 \pmod{11^{n+1}}.$$

#### 4.1. We now define

$$\eta(\tau) = e^{\pi i \tau / 12} f(x) \quad (\text{Im } \tau > 0),$$

where

$$f(x) = \prod_{r=1}^{\infty} (1 - x^r) \quad \text{and} \quad x = e^{2\pi i \tau}, \quad (29)$$

and

$$\phi(\tau) = \eta(121\tau)/\eta(\tau) = x^5 f(x^{121})/f(x), \quad \Phi(\tau) = 1/\phi(\tau). \quad (30)$$

We also let

$$\left. \begin{aligned} l_{2n-1} &= (13 \cdot 11^{2n-1} + 1)/24 \quad (n \geq 1), \\ l_{2n} &= (23 \cdot 11^{2n} + 1)/24 \quad (n \geq 1), \end{aligned} \right\} \quad (31)$$

so that  $l_n$  is the least positive integral solution of  $24l_n \equiv 1 \pmod{11^n}$ .

Further let

$$\left. \begin{aligned} \Lambda_{2n-1}(x) &= f(x^{11}) \sum_{m=0}^{\infty} p(11^{2n-1}m + l_{2n-1})x^{m+1}, \\ \Lambda_{2n}(x) &= f(x) \sum_{m=0}^{\infty} p(11^{2n}m + l_{2n})x^{m+1}, \end{aligned} \right\} \quad (n \geq 1), \quad (32)$$

and define a sequence of functions  $L_n(\tau)$  by

$$\left. \begin{aligned} L_1(\tau) &= U\phi(\tau), \\ L_{2n}(\tau) &= UL_{2n-1}(\tau) \quad (n \geq 1), \\ L_{2n+1}(\tau) &= \{U\phi(\tau L_{2n}(\tau))\} \quad (n \geq 1). \end{aligned} \right\} \quad (33)$$

We shall prove by induction that, for  $n \geq 1$ ,  $\Lambda_n(x)$  is the FS of  $L_n(\tau)$ . We have

$$U\{F_1(11\tau)F_2(\tau)\} = F_1(\tau)UF_2(\tau).$$

Now the FS of  $\phi(\tau)$  is

$$x^5 f(x^{121}) \sum_{m=0}^{\infty} p(m)x^m,$$

so that the FS of  $U\phi(\tau)$  is

$$f(x^{11}) \sum_{m=0}^{\infty} p(11m+6)x^{m+1} = \Lambda_1(x).$$

Assuming that the FS of  $L_{2n-1}(\tau)$  is  $\Lambda_{2n-1}(x)$ , we see that the FS of  $L_{2n}(\tau) = UL_{2n-1}(\tau)$  is

$$f(x) \sum_{m=0}^{\infty} p\{11^{2n-1}(11m+10) + l_{2n-1}\}x^{m+1} = \Lambda_{2n}(x).$$

Finally if the FS of  $L_{2n}(\tau)$  is  $\Lambda_{2n}(x)$ , then the FS of  $L_{2n+1}(\tau) = U\{\phi(\tau)L_{2n}(\tau)\}$  is

$$f(x^{11}) \sum_{m=0}^{\infty} p\{11^{2n}(11m+6) + l_{2n}\}x^{m+1} = \Lambda_{2n+1}(x).$$

Since the expansions of  $1/f(x)$  and  $1/f(x^{11})$  have integral coefficients with leading terms unity, Theorem 2 is equivalent to

**LEMMA 10.** *The FS of  $11^{-n}L_n(\tau)$  has integral coefficients.*



4.2. Since  $\phi(\tau)$  is not a function on  $\Gamma_0(11)$ , but on  $\Gamma_0(121)$ , we cannot apply the methods of §§2 and 3 immediately. However we do have

LEMMA 11. If  $F(\tau) \in S^\infty$ , then

(i)  $U\{\phi(\tau)F(\tau)\} \in S^\infty$ ,

(ii) the principal part of the expansions of  $U\{\phi(\tau)F(\tau)\}$  in powers of  $x = e^{-2\pi i/11\tau}$  at its pole  $\tau = 0$  is the same as the principal part of the FS expansion of  $11^{-2}\Phi(\tau)F(-1/121\tau)$  in powers of  $x = e^{2\pi i\tau}$ .

Further (i) and (ii) hold in the special case  $F(\tau) = 1$ .

Lemma 11 is proved by Lehner [6, Theorem 8]; there are some misprints corrected in Lehner [7, page 178].

We now have, by Lemma 5,

$$11^2 U\{\phi(\tau)h_n(\tau)\} = \sum d_{nr}h_r(\tau), \quad (34)$$

where the  $d_{nr}$  are constants, and in fact zero if  $11\psi(r) < \psi(n)+5$  or  $r > 11n+5$ . Further the  $d_{nr}$  are uniquely determined by the fact that the FS of

$$\Phi(\tau)G_n(11\tau) - \sum_{r=2}^{11n+5} d_{nr}G_r(\tau)$$

has no terms in  $x^{-11n-5}, \dots, x^{-3}, x^{-2}$ . Hence

$$\pi(d_{nr}) \geq 0. \quad (35)$$

We have also, from (34),

$$11^{2+\theta(n)} U\{\phi(\tau)g_n(\tau)\} = \sum d_{nr} 11^{\theta(r)} g_r(\tau),$$

and thus

$$\pi(d_{nr}) \geq \theta(n) - \theta(r) + 2. \quad (36)$$

We could, by using

$$\Phi(\tau)G_n(11\tau) \equiv G_5(\tau) \{G_n(\tau)\}^{11} \pmod{11},$$

obtain quite easily conditions under which  $\pi(d_{nr}) \geq 1$ . Unfortunately this is not quite enough to prove Theorem 1, and we require the following

LEMMA 12.

$$\Phi(\tau) \equiv G_5(\tau) + 11\{G_4(\tau) + 2G_3(\tau) + G_2(\tau) - 1 + 2g_2(\tau) + 3g_3(\tau) + g_4(\tau) + 5g_5(\tau)\} \pmod{11^2}.$$

This is proved in Appendix C.

We use the symbol  $(l, m)$  to denote an expression of the form

$$11 \sum_{i=1}^{m-1} \lambda_i G_i(\tau) + \sum_{i=m}^N G_i(\tau),$$

where  $l < m \leq N$  and the  $\lambda_i$  are integral constants. Then in terms also of the notation of §3.2, we have (using Tables 5 and 6)

$$\Phi(\tau)(k, l, m) \equiv (l_1, m_1) \pmod{11^2} \quad (m \geq 7), \quad (37)$$

where  $m_1 = m + 5$ ,  $l_1 = \min(l + 5, m - 5)$ .

Thus  $\Phi(\tau)G_2(11\tau) \equiv (14, 24) \pmod{11^2},$

$$\Phi(\tau)G_n(11\tau) \equiv (9n - 4, 9n + 6) \pmod{11^2} \quad (n \geq 3),$$

by (12) and (14). Hence

$$\pi(d_{nr}) \geq 2 \quad \text{if } r \leq 9n - 5. \quad (38)$$

We can now prove the result complementary to Lemma 7. We have

LEMMA 13. If  $F(\tau) \in Y$ , then  $11^{-1}U\{\phi(\tau)F(\tau)\} \in X$ .

*Proof.* We have, by (34),

$$11^{-1}U \sum \mu_n 11^{\eta(n)} \phi(\tau) g_n(\tau) = \sum \sum \mu_n 11^{\eta(n) - 3 - \theta(n) + \theta(r)} d_{nr} g_r(\tau).$$

Thus we have to show that, for all  $n$  and  $r$ ,

$$\eta(n) - 3 - \theta(n) + \theta(r) + \pi(d_{nr}) \geq \xi(r). \quad (39)$$

Since  $\pi(d_{nr}) \geq 0$ , (39) holds if  $\theta(r) - \xi(r) \geq \theta(n) - \eta(n) + 3$ . This is satisfied in the cases

$$n \geq 4, r \geq n + 10; \quad n = 2 \text{ or } 3, r \geq 12. \quad (40)$$

Also since  $\pi(d_{nr}) \geq \theta(n) - \theta(r) + 2$ , (39) holds if  $\eta(n) \geq \xi(r) + 1$ . This is satisfied in the cases

$$n \geq 4, r \leq n; \quad n = 3, r = 2. \quad (41)$$

Next, for  $n \geq 3$  and  $n + 1 \leq r \leq n + 9$ , we have  $\pi(d_{nr}) \geq 2$  and  $\theta(r) - \xi(r) \geq \theta(n) - \eta(n) + 1$ , which implies (39) for

$$n \geq 3, \quad n + 1 \leq r \leq n + 9. \quad (42)$$

Similarly we obtain

$$n = 2, \quad 4 \leq r \leq 11. \quad (43)$$

Finally we have by direct calculation  $\pi(d_{33}) = 3$ ,  $\pi(d_{22}) = 3$ ,  $\pi(d_{23}) = 4$ , which give (39) for

$$n = 2, r = 2 \text{ and } 3; \quad n = 3, r = 3. \quad (44)$$

Since (40) to (44) cover all integral  $n, r$  with  $n \geq 2, r \geq 2$ , Lemma 13 is proved. We have also

LEMMA 14. If  $F(\tau) \in Y^0$ , then  $11^{-1}U\{\phi(\tau)F(\tau)\} \in X^0$ .

We have

$$\lambda_2 = \sum \mu_n 11^{\eta(n) - 3 - \theta(n) + 2} d_{n2} = \sum \sigma_n, \quad \text{say.}$$

Now  $\pi(d_{n2}) \geq \theta(n) - \theta(2) + 2$ , so that for  $n \geq 4$ ,  $\eta(n) \geq 2$ , and so  $\pi(\sigma_n) \geq 1$ . For  $n = 3$ ,

$\pi(d_{32}) = 4$  by direct calculation, and so  $\pi(\sigma_3) = 1$ . Hence

$$\lambda_2 \equiv \mu_2 11^{-3} d_{22} \pmod{11}.$$

But  $\pi(d_{22}) = 3$ , and hence if  $\pi(\mu_2) = 0$ , then  $\pi(\lambda_2) = 0$ . This proves the lemma.

### 5. Proof of Theorem 1.

Using the remark at the end of Lemma 11 we find

$$L_1(\tau) = U\phi(\tau) = 11g_2(\tau) + 2 \cdot 11^2 g_3(\tau) + 11^3 g_4(\tau) + 11^4 g_5(\tau).$$

Hence  $11^{-1}L_1(T) \in X^0$ . It is now easily seen, by using the definition of  $L_n(\tau)$  in (33) and Lemmas 8 and 14, that

$$11^{1-2n}L_{2n-1}(\tau) \in X^0, \quad 11^{-2n}L_{2n}(\tau) \in Y^0.$$

This proves Lemma 10 and so Theorem 1. In addition we see, as in §3.5, that Theorem 1 is best possible in the sense that

$$p(l_n) \not\equiv 0 \pmod{11^{n+1}}.$$

It is clear that the inductions used to prove Theorem 2 are dominated by the values of  $\pi(c_{22})$  and  $\pi(d_{22})$ , in the sense that were either of these greater we could with greater effort establish a congruence modulo  $11^{\lfloor 3n/2 \rfloor}$  or thereabouts. The actual computed values of  $\pi(c_{nr})$  and  $\pi(d_{nr})$  are much larger than those given by our inequalities as is shown by Tables 7 and 8. The difficult part of the induction, apart from "accidental" low values of  $n$  and  $r$ , is when  $r$  is close to  $n$ , and in fact it seems certain that  $\pi(c_{nr})$  and  $\pi(d_{nr})$  are about equal to  $n$  in this case, not merely 3 or 4 as we prove. The introduction of a basis  $g_n(\tau)$  with different orders of zeros at  $\tau = i\infty$  is needed to cope with the case when  $r < n$ ; the numbers  $\theta(n) \doteq 6n/5$  which this involves are an inevitable and not wholly desirable complication. For  $r \gg n$  Lehner's basis is equally satisfactory. Finally, the actual classes of functions  $X, Y$  suffice for the induction, and are not best possible. We could use  $\pi(d_{nr}) \geq 1$  only, and a more elaborate form of Lemma 6, plus a good deal of actual computation for low values of  $n$  and  $r$ . This would avoid the appeal to Fine's equation, but the present method is shorter.

We may observe finally that, in comparison with  $q = 5$  and  $q = 7$ , this proof is indeed "langweilig", as Watson suggested. In those cases, we can in effect deal directly with  $Ug^n(\tau)$  at  $\tau = i\infty$ , using the modular equation. In fact, his actual induction can be reduced† to about 2 pages each for  $q = 5$  and  $q = 7$ , if it is expressed in terms of  $\pi(c_{nr})$  rather than fully written out formulae, by using explicit inequalities of the type  $\pi(c_{nr}) \geq [(5n-r+1)/2]$  for  $q = 5$ . I think it likely that in the present case  $q = 11$  there exists an inequality

$$\pi(c_{nr}) \geq [(11n-r+\delta)/10],$$

where  $\delta = \delta(n, r)$  is small and of irregular behaviour, but I can at present see no technique for establishing this.

† See A. O. L. Atkin, Ramanujan congruences for  $p_k(n)$ ; to appear in *Canadian J. Math.*

## APPENDIX A

**Proof of Lemma 3.** Following Newman [10], we define

$$\sum_{n=0}^{\infty} p_r(n)x^n = f^r(x), \quad (45)$$

where

$$f(x) = \prod_{m=1}^{\infty} (1-x^m).$$

We shall in this appendix, where no confusion can arise, write  $F(x)$  for the Fourier series of  $F(\tau)$ , with  $x = e^{2\pi i\tau}$ . If now functions  $g_2(x)$ ,  $g_3(x)$ ,  $G_2(x)$ ,  $G_3(x)$  are defined by

$$\left. \begin{aligned} 10g_2(x)f^5(x) &= -\sum_{n=0}^{\infty} \left\{ 1 + \left( \frac{n-3}{11} \right) \right\} p_5(n)x^n + 11^2 x^{25} f^5(x^{121}), \\ 14\{g_3(x) + g_2(x)\}f^7(x) &= -\sum_{n=0}^{\infty} \left\{ 1 + \left( \frac{2-n}{11} \right) \right\} p_7(n)x^n + 11^3 x^{35} f^7(x^{121}), \\ \{11^2 + 10G_2(x)\}f^5(x^{11}) &= \sum_{n=-2}^{\infty} p_5(11n+25)x^n, \\ \{11^3 + 14G_3(x) + 154G_2(x)\}f^7(x^{11}) &= \sum_{n=-3}^{\infty} p_7(11n+35)x^n, \end{aligned} \right\} \quad (46)$$

it follows from (2.5.2), (2.7), and (2.8) of [10] that  $G_2(\tau)$ ,  $G_3(\tau)$ ,  $g_2(\tau)$ ,  $g_3(\tau)$  belong to  $S$ , and that

$$G_2(-1/11\tau) = 11^2 g_2(\tau), \quad G_3(-1/11\tau) = 11^3 g_3(\tau). \quad (47)$$

By examination of the actual expansions in Table 1 we see that in fact  $G_2(\tau)$ ,  $G_3(\tau) \in S^0$  and  $g_2(\tau)$ ,  $g_3(\tau) \in S^\infty$ . We define next

$$B(\tau) = G_2(\tau)g_2(\tau) - 12. \quad (48)$$

$B(\tau)$  belongs to  $S$ , has a simple pole residue 1 at  $\tau = 0$  and  $\tau = i\infty$ , and satisfies

$$B(\tau) = B(-1/11\tau). \quad (49)$$

Since  $G_3(\tau)g_3(\tau)$  has the same properties, it follows that

$$G_3(\tau)g_3(\tau) = B(\tau) + \text{constant} = B(\tau) + 11. \quad (50)$$

We now define

$$\left. \begin{aligned} G_4(\tau) &= G_2^2(\tau) - 11G_3(\tau), & g_4(\tau) &= g_2^2(\tau) - g_3(\tau), \\ G_6(\tau) &= G_2(\tau)G_4(\tau), & g_6(\tau) &= g_2(\tau)g_4(\tau), \\ G_5(\tau) &= \eta^{12}(\tau)/\eta^{12}(11\tau), & g_5(\tau) &= \eta^{12}(11\tau)/\eta^{12}(\tau). \end{aligned} \right\} \quad (51)$$



That  $G_5(\tau) \in S^0$ ,  $g_5(\tau) \in S^\infty$  follows from Newman [10, (2,3,3)]. We have

$$\left. \begin{aligned} G_4(-1/11\tau) &= 11^4 g_4(\tau), \\ G_6(-1/11\tau) &= 11^6 g_6(\tau), \\ G_5(-1/11\tau) &= 11^6 g_5(\tau). \end{aligned} \right\} \quad (52)$$

Since  $G_5(\tau) - G_2(\tau)G_3(\tau) + 11G_4(\tau)$  has a pole of order  $m \leq 1$  at  $\tau = i\infty$ , and is zero at  $\tau = 0$ , it must be zero, since  $\Gamma_0(11)$  has genus 1. Hence

$$G_5(\tau) = G_2(\tau)G_3(\tau) - 11G_4(\tau). \quad (53)$$

We use this technique to derive the multiplication tables in Appendix B.

Next, we define inductively for  $n \geq 7$ ,

$$G_n(\tau) = G_{n-5}(\tau)G_5(\tau), \quad g_n(\tau) = g_{n-5}(\tau)g_5(\tau). \quad (54)$$

These results, together with the initial expansions in Table 1, establish the whole of Lemma 3 except for the assertions that the FS of  $G_n(\tau)$ ,  $g_n(\tau)$  have integral coefficients (they clearly have rational coefficients from (46)). These can be proved in various ways, of which we choose the following. The functions  $\alpha(\tau)$ ,  $\beta(\tau)$  of Fine [11, (3.20)], clearly have integral coefficients and belong to  $S^0$ . We thus can conclude that

$$G_2(\tau) = \alpha(\tau), \quad G_3(\tau) = \beta(\tau) - 3\alpha(\tau), \quad (55)$$

so that  $G_2(\tau)$ ,  $G_3(\tau)$ , and hence  $G_4(\tau)$ ,  $G_6(\tau)$ , have integral FS. It is also clear that  $G_5(\tau)$  and  $g_5(\tau)$  have integral FS. Now

$$g_2(\tau) = g_5(\tau)G_4(\tau), \quad g_3(\tau) = g_5(\tau)G_3(\tau), \quad (56)$$

so that  $g_2(\tau)$ ,  $g_3(\tau)$ , and hence  $g_4(\tau)$ ,  $g_6(\tau)$  have integral FS. The result for all  $n$  now follows from the definition (54).

## APPENDIX B

### Fourier Series Expansions

Table 1

With

$$x = e^{2\pi i\tau} \quad \text{and} \quad F(\tau) = \sum_{r=N}^{\infty} \alpha_r x^r,$$

we write

$$F(\tau) = x^N(\alpha_N, \alpha_{N+1}, \alpha_{N+2}, \dots).$$

$$\begin{aligned} B(\tau) &= x^{-1}(1, 5, 17, 46, 116, 252, 533, 1034, 1961, \dots), \\ g_2(\tau) &= x(1, 5, 19, 63, 185, 502, 1270, 3046, 6968, 15335, \dots), \\ g_3(\tau) &= x^2(1, 9, 49, 214, 800, 2685, 8274, 23829, 64843, \dots), \\ g_4(\tau) &= x^3(1, 14, 102, 561, 2563, 10285, 37349, 125290, \dots), \\ g_5(\tau) &= x^5(1, 12, 90, 520, 2535, 10908, 42614, 153960, \dots), \\ g_6(\tau) &= x^4(1, 19, 191, 1400, 8373, 43277, 199982, 844734, \dots), \\ G_2(\tau) &= x^{-2}(1, 2, -12, 5, 8, 1, 7, -11, 10, -12, \dots), \\ G_3(\tau) &= x^{-3}(1, -3, -5, 24, -13, -22, 13, -5, 51, \dots), \\ G_4(\tau) &= x^{-4}(1, -7, 13, 17, -84, 57, 93, -81, -63, \dots), \\ G_5(\tau) &= x^{-5}(1, -12, 54, -88, -99, 540, -418, -648, 594, \dots), \\ G_6(\tau) &= x^{-6}(1, -5, -13, 132, -233, -305, 1404, -910, -1533, \dots). \end{aligned}$$

Seq 3295 = N 1586.5  
3296 N 1594.5  
3297 N 1978.5

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757 Fourier series  
758  
5759 coeff of  $\frac{1}{x}$   
760  
761 model function  
762  
..., ~~5762~~ ~~755~~ 735  
1533, ...). 5764

$$\begin{aligned} G_4 &= G_2^2 - 11G_3, & g_4 &= g_2^2 - g_3, \\ G_5 &= G_3G_2 - 11G_4, & 11g_5 &= g_2g_3 - g_4, \\ G_6 &= G_2G_4, & g_6 &= g_2g_4, \\ G_{n+5} &= G_nG_5 \quad (n \geq 2), & g_{n+5} &= g_ng_5 \quad (n \geq 2). \end{aligned}$$
$$\begin{array}{lll} G_2(-1/11\tau) = 11^2 g_2(\tau), & G_2(\tau) = x^{-2} + \dots, & g_2(\tau) = x + \dots, \\ G_3(-1/11\tau) = 11^3 g_3(\tau), & G_3(\tau) = x^{-3} + \dots, & g_3(\tau) = x^2 + \dots, \\ G_4(-1/11\tau) = 11^4 g_4(\tau), & G_4(\tau) = x^{-4} + \dots, & g_4(\tau) = x^3 + \dots, \\ G_5(-1/11\tau) = 11^6 g_5(\tau), & G_5(\tau) = x^{-5} + \dots, & g_5(\tau) = x^5 + \dots, \\ G_6(-1/11\tau) = 11^6 g_6(\tau), & G_6(\tau) = x^{-6} + \dots, & g_6(\tau) = x^4 + \dots \end{array}$$
$$\begin{array}{ll} BG_2 = 11^2 + G_3, & Bg_2 = 1 + 11g_3, \\ BG_3 = 11G_2 - G_3 + G_4, & Bg_3 = g_2 - g_3 + 11g_4, \\ BG_4 = 11G_3 + G_5, & Bg_4 = g_3 + 11^2g_5, \\ BG_5 = -12G_5 + G_6, & Bg_5 = -12g_5 + g_6, \\ BG_6 = 11^2G_4 + 11G_5 + G_7 & Bg_6 = g_4 + 11g_5 + 11^2g_7. \end{array}$$

Multiplication Table 5

|       | $G_2$         | $G_3$         | $G_4$         | $G_6$                                    |
|-------|---------------|---------------|---------------|------------------------------------------|
| $G_2$ | $11G_3 + G_4$ | $11G_4 + G_5$ | $G_6$         | $11^2G_5 + 12G_7 + G_8$                  |
| $G_3$ |               | $-G_5 + G_6$  | $11G_5 + G_7$ | $11G_7 + 11G_8 + G_9$                    |
| $G_4$ |               |               | $G_7 + G_8$   | $11G_8 + 12G_9 + G_{10}$                 |
| $G_6$ |               |               |               | $11^2G_9 + 11G_{10} + 12G_{11} + G_{12}$ |

$$G_{n+5} = G_n G_5$$

Multiplication Table 6

|       | $g_2$                 | $g_3$                 | $g_4$                | $g_5$    | $g_6$                   |
|-------|-----------------------|-----------------------|----------------------|----------|-------------------------|
| $G_2$ | $B + 12$              | $1 + 11g_2$           | $g_2 + 11g_3$        | $g_4$    | $g_3 + 12g_4 + 11^2g_5$ |
| $G_3$ | $G_2 + 11$            | $B + 11$              | $1 + 11g_2$          | $g_3$    | $g_2 + 11g_3 + 11g_4$   |
| $G_4$ | $G_3 + G_2$           | $G_2 + 11$            | $B + 12$             | $g_2$    | $1 + 12g_2 + 11g_3$     |
| $G_5$ | $G_4$                 | $G_3$                 | $G_2$                | 1        | $B + 12$                |
| $G_6$ | $G_5 + 12G_4 + 11G_3$ | $G_4 + 11G_3 + 11G_2$ | $G_3 + 12G_2 + 11^2$ | $B + 12$ | $(B + 12)^2$            |

Tables 7 and 8 give the actual computed values of  $\pi(\gamma_{nr})$  and  $\pi(\delta_{nr})$  in

$$Ug_n(\tau) = \sum \gamma_{nr} g_r(\tau), \quad U\{\phi(\tau)g_n(\tau)\} = \sum \delta_{nr} g_r(\tau).$$

The calculations were performed modulo  $11^{10}$ , and  $T$  stands for " $\geq 10$ ".

Table 7.  $\pi(\gamma_{nr})$ 

|         | $r = 2$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10  |
|---------|---------|---|---|---|---|---|---|---|-----|
| $n = 2$ | 1       | 2 | 3 | 5 | 5 | 7 | 9 | 8 | $T$ |
| 3       | 1       | 3 | 3 | 5 | 5 | 7 | 7 | 8 | $T$ |
| 4       | 1       | 2 | 3 | 5 | 6 | 6 | 7 | 8 | $T$ |
| 5       | 1       | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $T$ |
| 6       | 1       | 2 | 4 | 5 | 4 | 6 | 7 | 8 | $T$ |
| 7       | 1       | 2 | 2 | 4 | 4 | 6 | 8 | 8 | $T$ |
| 8       | 1       | 1 | 2 | 4 | 4 | 6 | 7 | 8 | $T$ |
| 9       | 0       | 1 | 2 | 4 | 4 | 6 | 7 | 8 | $T$ |
| 10      | 0       | 1 | 2 | 4 | 6 | 6 | 7 | 8 | $T$ |

Table 8.  $\pi(\delta_{nr})$ 

| $r=$  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10  |
|-------|---|---|---|---|---|---|---|---|-----|
| $n=2$ | 1 | 2 | 2 | 4 | 4 | 7 | 7 | 9 | $T$ |
| 3     | 1 | 1 | 2 | 5 | 4 | 6 | 7 | 8 | $T$ |
| 4     | 0 | 1 | 2 | 4 | 4 | 6 | 7 | 8 | $T$ |
| 5     | 0 | 1 | 2 | 4 | 4 | 6 | 7 | 8 | $T$ |
| 6     | 0 | 1 | 3 | 4 | 5 | 6 | 7 | 8 | $T$ |
| 7     | 0 | 1 | 2 | 4 | 4 | 6 | 8 | 8 | $T$ |
| 8     | - | 1 | 2 | 4 | 4 | 6 | 7 | 9 | $T$ |
| 9     | - | 1 | 2 | 4 | 4 | 6 | 7 | 8 | $T$ |
| 10    | - | 1 | 2 | 3 | 4 | 6 | 7 | 9 | $T$ |

Table 9

This table shows the relation of the notations of Lehner [6, 7], Fine [11], and Atkin and Hussain [12] to that of this paper.

|              | Lehner              | Fine                          | Atkin and Hussain      |
|--------------|---------------------|-------------------------------|------------------------|
| $B(\tau)$    | $A(\tau) - 11$      |                               |                        |
| $g_2(\tau)$  | $C(\tau)$           |                               |                        |
| $g_3(\tau)$  | $D(\tau) - C(\tau)$ |                               |                        |
| $\phi(\tau)$ | $\Phi(\tau)$        | $u^{-1}(11\tau)$              |                        |
| $G_2(\tau)$  |                     | $\alpha(\tau)$                | $-\lambda - 13$        |
| $G_3(\tau)$  |                     | $\beta(\tau) - 3\alpha(\tau)$ | $-\mu + 6\lambda + 16$ |
| $G_5(\tau)$  |                     | $v(\tau)$                     |                        |
| $L_n(\tau)$  | $L(\tau; 11^n)$     |                               |                        |

## APPENDIX C

**Proof of Lemma 12.** The modular equation of degree 11 in  $\Phi(\tau/11)$  with coefficients in  $\mathbb{S}$  is given by Fine [11, (3.21)]. If we subject this to the transformation  $\tau \rightarrow -1/11\tau$ , and observe that  $\Phi(-1/121\tau) = 11\phi(\tau)$ , we obtain in our notation (the argument  $\tau$  being omitted for brevity)



