AS 206

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"CURIOUSER AND CURIOUSER" SAID ALICE. FURTHER REFLECTIONS ON AN INTERESTING RECURSIVE FUNCTION

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... relationship between a previously investigated recursive function and the

white, and then a red cow; and so on. What was the colour of the nth cow?"

Mulcrone's solution [3] is as follows:

n

[illegible] straightforward but tedious and are therefore

Table 5 Examples of normalised Fibary representations

n	*Normalised Fibary representation for n*
0	00
1	10
2	100
3	1000
4	1010
5	10000

[redacted]

Notes

[redacted]

The proof proceeds by addressing the three possible structures for k described above. The proof in each case is similar so only one case is treated below.

Consider Case (iii) above:

If k has the normalised Fibary representation $z000$ then $k+1$ has the normalised

Fibary representation $z010$.

Since $x0 = z000$ it follows that $x = z00$ and $x00 = z0000$.

Thus $z000 + z00 = z0000$ by the induction hypothesis.

To show that the result is true for $k+1$, first note that $z010 = z000 + 10$.

Then

$$\begin{aligned} z010 + z01 &= (z000 + 10) + (z00 + 1) && \text{... Fibary arithmetic} \\ &= (z000 + z00) + (10 + 1) && \text{... by commutativity of } + \\ &= z0000 + 100 && \text{... by induction hypothesis} \\ &= z0100 \end{aligned}$$

5. AN ALTERNATIVE DEFINITION of h

DEFINITION Let **G** be defined recursively by:

$$\mathbf{G}: \mathbb{N} \to \mathbb{N}$$

$$\mathbf{G}(0) = 0$$

$$\mathbf{G}(k) = \mathbf{Fib}(m-1) + \mathbf{G}(k - \mathbf{Fib}(m))$$
$$\text{where } \mathbf{Fib}(m) \leq k < \mathbf{Fib}(m+1).$$

Let $\mathbf{F}: \mathbb{N} \to \mathbb{N}$ be defined by:

$\mathbf{F}(n)$ = the number with Fibary representation that is the normalised Fibary representation of n with the last 0 removed.

e.g. $\mathbf{F}(7) = \mathbf{F}(10100) = 1010 = 4$

$\mathbf{F}(20) = \mathbf{F}(1010100) = 101010 = 12$

THEOREM 2 *F satisfies the definition of* **G**.

Proof By induction

ii) Induction step:

Assume the result for all $n<k$

$$\mathbf{G}(k)=\mathbf{Fib}(m-1)+\mathbf{G}(k-\mathbf{Fib}(m)) \text{ where } \mathbf{Fib}(m)\leqq k<\mathbf{Fib}(m+1)$$

k has the normalised Fibary representation of $10x0$ (where x has $m-3$ digits)

$$\mathbf{G}(k)=\mathbf{Fib}(m-1)+\mathbf{G}(x0).$$

Since $x<k$ it follows by the induction hypothesis that $\mathbf{G}(x0)=x$.
Then $\mathbf{G}(k)=\mathbf{Fib}(m-1)+x=10x$.
$\mathbf{F}(10x0)=10x$.

THEOREM 3 *The functions* $\mathbf{F}$ *and* $\mathbf{h}$ *are equivalent.*

Proof By induction

i) Base step:

This follows directly from the definitions of $\mathbf{F}$ and $\mathbf{h}$.

$$\mathbf{F}(0)=0=\mathbf{h}(0)$$

$$\mathbf{F}(1)=1=\mathbf{h}(1).$$

ii) Induction step:

Assume that $\mathbf{F}(j)=\mathbf{h}(j)$ for all $j\leqq k$.
$\mathbf{h}(k+1)=k+1-\mathbf{h}(\mathbf{h}(k))$ ... from the definition of $\mathbf{h}$.
By the hypothesis $\mathbf{h}(k)=\mathbf{F}(k)$ and, since $\mathbf{h}(k)\leqq k$,

$$\mathbf{h}(\mathbf{h}(k))=\mathbf{F}(\mathbf{F}(k))$$

$$\mathbf{h}(k+1)=k+1-\mathbf{F}(\mathbf{F}(k)).$$

However, k has one of the following normalised representations

a) $x00(10)^y$
b) $x00(10)^y0$
c) $x000$.

Consider each of these cases in turn

Case (a)

$\mathbf{F}(k)=x00(10)^{y-1}1$	... definition of $\mathbf{F}$
$=x01(00)^{y-1}0$	... normalisation
$\mathbf{F}(\mathbf{F}(k))=x01(00)$	... definition of $\mathbf{F}$
$\mathbf{h}(k+1)=x01(00)-x01(00)$	... successor
$=x01(00)^{y-1}0$	... subtraction
$=\mathbf{F}(k+1).$	

Case (b)

$$\begin{aligned}
\mathbf{F}(k) &= x00(10)^y && \ldots \text{definition of } \mathbf{F}\\
\mathbf{F}(\mathbf{F}(k)) &= x00(10)^{y-1}1 && \ldots \text{definition of } \mathbf{F}\\
&= x01(00)^{y-1}0 && \ldots \text{normalisation}\\
\mathbf{h}(k+1) &= x01(00)^y 0 - x01(00)^{y-1}0 && \ldots \text{successor}\\
&= x01(00) && \ldots \text{subtraction}\\
&= \mathbf{F}(k+1).
\end{aligned}$$

Case (c)