

Beauville Periods in OEIS

Periods Converter

See also: <https://community.wolfram.com/groups/-/m/t/2926119>

In[1]:= \$WPFCs = { ... } ▢

Out[1]=
$$\begin{aligned} & \left\{ -21 G2[0] G2[1]^3 G3[0] + 18 G2[0]^2 G2[1]^2 G3[1] + \right. \\ & \quad 8 G2[0]^3 G2[2] \times G3[1] - 216 G2[2] G3[0]^2 G3[1] + 108 G2[1] \times G3[0] G3[1]^2 - \\ & \quad 120 G2[0] G3[1]^3 - 8 G2[0]^3 G2[1] \times G3[2] + 216 G2[1] G3[0]^2 G3[2], \\ & \quad -16 \left(9 G2[0]^2 G2[1]^2 G3[0] - 3 G2[0]^3 G2[2] \times G3[0] + \right. \\ & \quad \quad 81 G2[2] G3[0]^3 - 7 G2[0]^3 G2[1] \times G3[1] - 135 G2[1] G3[0]^2 G3[1] + \\ & \quad \quad 108 G2[0] \times G3[0] G3[1]^2 + 2 G2[0]^4 G3[2] - 54 G2[0] G3[0]^2 G3[2] \left. \right), \\ & \quad \left. 16 \left(G2[0]^3 - 27 G3[0]^2 \right) \left(-3 G2[1] \times G3[0] + 2 G2[0] \times G3[1] \right) \right\} \end{aligned}$$

```
In[3]:= EllipticPeriodsAnnihilator[
  g2x_, g3x_] := With[{res0 = ReplaceAll[
    $WPFCs, {
      G2[n_] -> D[g2x, {x, n}],
      G3[n_] -> D[g3x, {x, n}]
    }
  ]},
  Factor[Cancel[Divide[#, PolynomialGCD@@#
    ] &[Together[res0]]
  ]]
```

After Herfurtner 1991

<https://link.springer.com/article/10.1007/BF01445211>

1,1,1,9 * ~ A006077 ~ BI, BVI

$$\begin{array}{ll}
 I_1 I_1 I_1 I_9 & G_2 = 3X(9X^3 - 8Y^3) \\
 (1, \eta, \eta^2, \infty) & G_3 = 27X^6 - 36X^3Y^3 + 8Y^6 \\
 & \Delta = 2^6 \cdot 3^3 Y^9 (X^3 - Y^3) \\
 & \mathcal{J} = \frac{1}{64} \frac{X^3(9X^3 - 8Y^3)^3}{Y^9(X^3 - Y^3)} \\
 & \text{CR} = -\eta, \quad \eta = e^{\frac{2\pi i}{3}}
 \end{array}$$

```
In[12]:= data = {
  3 x (9 x^3 - 8 y^3),
  27 x^6 - 36 x^3 y^3 + 8 y^6
} /. y -> 1 /. x -> -9 x + 1
```

```
Out[12]= {3 (-8 + 9 (1 - 9 x)^3) (1 - 9 x), 8 - 36 (1 - 9 x)^3 + 27 (1 - 9 x)^6}
```

```
In[13]:= ode = EllipticPeriodsAnnihilator @@ data
```

```
Out[13]= {3 (-1 + 9 x), (-1 + 9 x)^2, x (1 - 9 x + 27 x^2)}
```

proof:

```
In[14]:= gf = 1 / (1 - 3 x) Hypergeometric2F1[1 / 3, 2 / 3, 1, ((-3 x) / (1 - 3 x))^3];
```

```
In[15]:= FullSimplify[Dot[ode, D[gf, {x, #}] & /@ {0, 1, 2}]]
```

```
Out[15]= 0
```

```
In[16]:= CoefficientList[Series[gf, {x, 0, 10}], x]
```

```
Out[16]= {1, 3, 9, 21, 9, -297, -2421, -12933, -52407, -145293, -35091}
```

Another proof:

```
ode = Factor[9 hypergeometric /. {a -> 1/3, b -> 2/3, c -> 1}] == 0
ode2 = ODEChangeVariables[ode, {T}, x, z -> ((-3 x) / (1 - 3 x))^3 ["Result"];
ode3 = ode2 /. Subscript[T, 1] -> Function[z, (1 - 3 z) G[z]];
final =
  Divide[Factor[Coefficient[-ode3[[1]], D[G[x], {x, #}]] & /@ {0, 1, 2}], (3 x - 1)^3]
```

Out[*]=

$$2 T[z] - 9 T'[z] + 18 z T'[z] - 9 z T''[z] + 9 z^2 T''[z] == 0$$

Out[*]=

$$\{3(-1 + 9x), (-1 + 9x)^2, x(1 - 9x + 27x^2)\}$$

1,1,2,8 * ~ A081085 ~ BII, BV

$$\begin{aligned}
 I_1 I_1 I_2 I_8 & \quad G_2 = 3(16X^4 - 16X^2Y^2 + Y^4) \\
 (-1, 1, 0, \infty) & \quad G_3 = 64X^6 - 96X^4Y^2 + 30X^2Y^4 + Y^6 \\
 & \quad \Delta = 2^2 \cdot 3^6 X^2 Y^8 (X+Y)(X-Y) \\
 & \quad \mathcal{J} = \frac{1}{108} \frac{(16X^4 - 16X^2Y^2 + Y^4)^3}{X^2 Y^8 (X+Y)(X-Y)} \\
 & \quad CR = -1
 \end{aligned}$$

```
In[19]:= data = {
  3 (16 x^4 - 16 x^2 y^2 + y^4),
  64 x^6 - 96 x^4 y^2 + 30 x^2 y^4 + y^6
} /. y -> 1 /. x -> -8 x + 1
```

Out[19]=

$$\{3(1 - 16(1 - 8x)^2 + 16(1 - 8x)^4), 1 + 30(1 - 8x)^2 - 96(1 - 8x)^4 + 64(1 - 8x)^6\}$$

```
In[21]:= ode = EllipticPeriodsAnnihilator @@ data
```

Out[21]=

$$\{4(-1 + 8x), 1 - 24x + 96x^2, x(-1 + 4x)(-1 + 8x)\}$$

proof:

```
In[22]:= gf = 4 / Pi / 2 EllipticK[16 x^2 / (1 - 4 x)^2] / (1 - 4 x)
```

Out[22]=

$$\frac{2 \operatorname{EllipticK}\left[\frac{16 x^2}{(1-4 x)^2}\right]}{\pi (1-4 x)}$$

```
In[23]:= FullSimplify[Dot[ode, D[gf, {x, #}] & /@ {0, 1, 2}]]
```

```
Out[23]=
```

0

```
In[24]:= CoefficientList[Series[gf, {x, 0, 10}], x]
```

```
Out[24]=
```

{1, 4, 20, 112, 676, 4304, 28496, 194240, 1353508, 9593104, 68906320}

1,2,3,6 * ~ A002893, A093388, A000172 ~BIV

$$\begin{aligned}
 I_1 I_2 I_3 I_6 & \quad G_2 = 12(X^4 - 4X^3Y + 2XY^3 + Y^4) \\
 (4, -\frac{1}{2}, 0, \infty) & \quad G_3 = 4(2X^6 - 12X^5Y + 12X^4Y^2 + 14X^3Y^3 + 3X^2Y^4 + 6XY^5 + 2Y^6) \\
 & \quad \Delta = 2^4 \cdot 3^6 X^3 Y^6 (2X + Y)^2 (X - 4Y) \\
 & \quad \mathcal{J} = \frac{4(X^4 - 4X^3Y + 2XY^3 + Y^4)^3}{27 X^3 Y^6 (2X + Y)^2 (X - 4Y)} \\
 & \quad CR = -8
 \end{aligned}$$

A002893

```
In[29]:= data = Factor[{
```

```
12 * (x^4 - 4 x^3 y + 2 x y^3 + y^4),
```

```
4 (2 x^6 - 12 x^5 y + 12 x^4 y^2 + 14 x^3 y^3 + 3 x^2 y^4 + 6 x y^5 + 2 y^6)
```

```
} /. y -> 1/4 + 1/4 x /. x -> -x]
```

```
Out[29]=
```

$$\left\{ \frac{3}{64} (1 + 3x) (1 - 15x + 75x^2 + 3x^3), -\frac{1}{512} (-1 + 6x + 3x^2) (1 - 12x + 30x^2 - 540x^3 + 9x^4) \right\}$$

```
In[30]:= ode = EllipticPeriodsAnnihilator @@ data
```

```
Out[30]=
```

$$\{-3(-1 + 3x), -1 + 20x - 27x^2, -((-1 + x)x(-1 + 9x))\}$$

$$\text{In[31]:= gf} = -\frac{4 \text{EllipticK}\left[\frac{(\sqrt{1+(-6+8\sqrt{x}-3x)}x - \sqrt{1-x(6+8\sqrt{x}+3x)})^2}{(\sqrt{1+(-6+8\sqrt{x}-3x)}x + \sqrt{1-x(6+8\sqrt{x}+3x)})^2}\right]}{-\pi \sqrt{-(1+\sqrt{x})^3(-1+3\sqrt{x})} - \pi \sqrt{-(-1+\sqrt{x})^3(1+3\sqrt{x})}};$$

```
In[32]:= FullSimplify[Dot[ode, D[gf, {x, #}] & /@ {0, 1, 2}]]
```

```
Out[32]=
```

0

```
In[33]:= CoefficientList[Series[gf, {x, 0, 10}], x]
```

```
Out[33]=
```

{1, 3, 15, 93, 639, 4653, 35169, 272835, 2157759, 17319837, 140668065}

A093388

```

In[34]:= data = Factor[{
      12 * (x^4 - 4 x^3 y + 2 x y^3 + y^4),
      4 (2 x^6 - 12 x^5 y + 12 x^4 y^2 + 14 x^3 y^3 + 3 x^2 y^4 + 6 x y^5 + 2 y^6)
    } /. y -> 1/4 + 1/4 x /. x -> -x /. x -> -8 x + 1]

Out[34]=
{ 12 (-1 + 6 x) (-1 + 18 x - 84 x^2 + 24 x^3), -8 (1 - 12 x + 24 x^2) (-1 + 24 x - 192 x^2 + 504 x^3 + 72 x^4) }

In[35]:= ode = EllipticPeriodsAnnihilator @@ data
Out[35]=
{-6 (-1 + 12 x), -1 + 34 x - 216 x^2, -x (-1 + 8 x) (-1 + 9 x) }

In[36]:= gf = Hypergeometric2F1[1/3, 2/3, 1, x^2 * (8 * x - 1) / (2 * x - 1/3)^3] / (1 - 6 * x);
In[37]:= FullSimplify[Dot[ode, D[gf, {x, #}] & /@ {0, 1, 2}]]
Out[37]=
0

In[38]:= CoefficientList[Series[gf, {x, 0, 10}], x]
Out[38]=
{1, 6, 42, 312, 2394, 18756, 149136, 1199232, 9729882, 79527084, 654089292}

```

A000172

```

In[39]:= data = Together[{
      12 * (x^4 - 4 x^3 y + 2 x y^3 + y^4),
      4 (2 x^6 - 12 x^5 y + 12 x^4 y^2 + 14 x^3 y^3 + 3 x^2 y^4 + 6 x y^5 + 2 y^6)
    } /. y -> 1/4 + 1/4 x /. x -> -x /. x -> -8/9 x + 1/9]

Out[39]=
{
  4
  ----- (1 - 8 x + 240 x^2 - 464 x^3 + 16 x^4),
  243
  -
  8 (1 - 12 x - 480 x^2 + 3080 x^3 - 12072 x^4 + 4128 x^5 + 64 x^6)
  -----
  19683
}

In[40]:= ode = EllipticPeriodsAnnihilator @@ data
Out[40]=
{-2 (1 + 4 x), 1 - 14 x - 24 x^2, -x (1 + x) (-1 + 8 x) }

In[41]:= gf = Hypergeometric2F1[1/3, 2/3, 1, 27 x^2 / (1 - 2 x)^3] / (1 - 2 x);
In[42]:= FullSimplify[Dot[ode, D[gf, {x, #}] & /@ {0, 1, 2}]]
Out[42]=
0

In[43]:= CoefficientList[Series[gf, {x, 0, 10}], x]
Out[43]=
{1, 2, 10, 56, 346, 2252, 15184, 104960, 739162, 5280932, 38165260}

```

1,1,5,5 ~ A005258 ~ BIII

$$\begin{aligned}
I_1 I_1 I_5 I_5 & \quad G_2 = 3(X^4 - 12X^3Y + 14X^2Y^2 + 12XY^3 + Y^4) \\
(\omega_1, \omega_2, 0, \infty) & \quad G_3 = X^6 - 18X^5Y + 75X^4Y^2 + 75X^2Y^4 + 18XY^5 + Y^6 \\
& \quad \Delta = 2^6 \cdot 3^6 X^5 Y^5 (X^2 - 11XY - Y^2) \\
& \quad \mathcal{J} = \frac{1}{2^6 \cdot 3^3} \frac{(X^4 - 12X^3Y + 14X^2Y^2 + 12XY^3 + Y^4)^3}{X^5 Y^5 (X^2 - 11XY - Y^2)} \\
\text{CR} &= \left(\frac{1 + \sqrt{5}}{1 - \sqrt{5}} \right)^5, \quad \omega_{1,2} = \left(\frac{1 \pm \sqrt{5}}{2} \right)^5
\end{aligned}$$

```
In[44]:= data = {
  3 * (x^4 - 12 x^3 * y + 14 x^2 y^2 + 12 x * y^3 + y^4),
  x^6 - 18 x^5 * y + 75 * x^4 * y^2 + 75 * x^2 y^4 + 18 * x * y^5 + y^6
} /. y -> 1 /. x -> -x
```

```
Out[44]= { 3 (1 - 12 x + 14 x^2 + 12 x^3 + x^4), 1 - 18 x + 75 x^2 + 75 x^4 + 18 x^5 + x^6 }
```

```
In[45]:= ode = EllipticPeriodsAnnihilator @@ data
Factor[CoefficientList[ode[[1]], x]]
Factor[CoefficientList[D[ode[[3]], x] - ode[[2]], x]]
```

```
Out[45]= { 3 + x, -1 + 22 x + 3 x^2, x (-1 + 11 x + x^2) }
```

```
Out[46]= { 3, 1 }
```

```
Out[47]= { }
```

```
In[48]:= gf = Divide[Hypergeometric2F1[1/12, 5/12, 1,
  1728 * x^5 * (1 - 11 * x - x^2) / (1 - 12 * x + 14 * x^2 + 12 * x^3 + x^4)^3
], (1 - 12 * x + 14 * x^2 + 12 * x^3 + x^4)^(1/4)]
```

```
Out[48]= Hypergeometric2F1[1/12, 5/12, 1, 1728 x^5 (1 - 11 x - x^2) / (1 - 12 x + 14 x^2 + 12 x^3 + x^4)^3] / (1 - 12 x + 14 x^2 + 12 x^3 + x^4)^(1/4)
```

```
In[49]:= FullSimplify[Dot[ode, D[gf, {x, #}] & /@ {0, 1, 2}]]
```

```
Out[49]= 0
```

```
In[50]:= CoefficientList[Series[gf, {x, 0, 10}], x]
```

```
Out[50]= {1, 3, 19, 147, 1251, 11253, 104959, 1004307, 9793891, 96918753, 970336269}
```

Acknowledgements

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Author

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See also

Don Zagier, “Integral solutions...” (2009).

<https://people.mpim-bonn.mpg.de/zagier/files/tex/AperylikeRecEqs/fulltext.pdf>

$$\text{Sequence A: } u_n = \sum_{k=0}^n \binom{n}{k}^3 = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{n},$$

$$\text{Sequence B: } u_n = \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \binom{3k}{k} \binom{2k}{k},$$

$$\text{Sequence C: } u_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k},$$

$$\text{Sequence D: } u_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k}^2,$$

$$\text{Sequence E: } u_n = \sum_{k=0}^{\lfloor n/2 \rfloor} 4^{n-2k} \binom{n}{2k} \binom{2k}{k}^2.$$

The formulas for the sequences **B** and **E** say that the corresponding generating functions $\sum u_n t^n$ have hypergeometric representations:

$$F_{\mathbf{B}}(t) = \frac{1}{1-3t} F\left(\frac{1}{3}, \frac{2}{3}; 1; \left(\frac{-3t}{1-3t}\right)^3\right), \quad F_{\mathbf{E}}(t) = \frac{1}{1-4t} F\left(\frac{1}{2}, \frac{1}{2}; 1; \left(\frac{4t}{1-4t}\right)^2\right).$$

nor (since the elliptic surfaces have non-trivial automorphisms) need they all be different. The actual correspondence turns out to be as follows:

Beauville Family:	IV	I, VI	III	II, V
Picard-Fuchs equation:	A, C, F	B, #14	D	E, G, #11

One way to see this is to compute the j -invariant of each of Beauville's family as a rational function of t , invert this equation to compute t as an explicit modular function, and compare with the list of t 's which we found before. The j -invariant in turn is most conveniently found by rewriting Beauville's equations in Weierstrass form $y^2 = f(x)$ (f a polynomial of degree 3). For the reader's convenience, we give the transformation needed in each of the 6 cases do this, and the resulting formula for the j -invariant:

	Transformation to Weierstrass form	j -invariant
I	$X, Y = t(x \pm y), Z = 2(1 + 3x)$	$-t^3(t^3 - 216)^3 / (t^3 + 27)^3$
II	$X = xt, Y = x + y, Z = t$	$2^8(t^4 - t^2 + 1)^3 / t^4(t^2 - 1)^2$
III	$X = 1 + (y + tx)/(1 - x), Y = 2x, Z = 2$	$(t^4 - 12t^3 + 14t^2 + 12t + 1)^3 / t^5(t^2 - 11t - 1)$
IV	$X, Y = 1 + tx \pm y, Z = 2x(1 + tx)$	$(t + 2)^3(t^3 + 6t^2 - 12t + 8)^3 / t^3(t - 1)^2(t + 8)$
V	$X, Y = 1 + tx \pm y, Z = 2x(1 + tx)$	$(t^4 + 16t^2 + 16)^3 / t^2(t^2 + 16)$
VI	$X = 2x, Y = y - xt - 1, Z = 2x^2$	$-t^3(t^3 + 24)^3 / (t^3 + 27)$