

EXPANSION OF THE NUMBERS BY UNIT FRACTIONS

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The following is an English translation, by Georg Fischer [1], of Friedrich Engel's speech: *Entwicklung der Zahlen nach Stammbrüchen*, Verhandlungen der 52. Versammlung Deutscher Philologen und Schulmänner, 1913, Marburg, pp. 190–191.

Wednesday, October 1st, 9 o'clock in the morning.

[List of participants ...]

Thereafter Prof. Dr. Engel (Gießen) rose to speak about Expansion of the numbers by unit fractions. The speaker explains:

For each positive number α there is a uniquely defined series expansion

$$\alpha = a + \frac{1}{q_1} + \frac{1}{q_2} + \cdots,$$

where $a, q_1, q_2, \dots$ represent integer numbers and where $a < \alpha \leq a + 1$, while the numbers $q_1, q_2, \dots$ are determined iteratively by the requirement that always

$$a + \frac{1}{q_1} + \cdots + \frac{1}{q_n} < \alpha \leq a + \frac{1}{q_1} + \cdots + \frac{1}{q_{n-1}} + \frac{1}{q_n - 1}$$

must hold. One finds that $q_{v+1} > q_v^2 - q_v$ must hold and that vice versa each infinite series of the form shown above which fulfills this requirement is convergent. A number α is rational if and only if, beginning at a certain q_n , always

$$q_{n+v+1} = q_{n+v}^2 - q_{n+v}$$

holds.

In the same way can be developed:

$$\alpha = a + \frac{1}{q_1} + \frac{1}{q_1 q_2} + \cdots + \frac{1}{q_1 q_2 \cdots q_n} + \cdots.$$

Now α is rational if and only if, beginning at a certain q_n , always $q_{n+v+1} = q_{n+v}$ holds. For e this leads to the known series expansion, and at the same time to a simple proof of the irrationality of e . By the way the same holds for each power $e^{\frac{1}{v}}$, where v is a positive integer number.

Georg Cantor remarked already in 1869 in the "Zeitschrift für Mathematik und Physik" [2] that each positive number $\alpha > 1$ allows for a uniquely defined product expansion

$$\alpha = a \left(1 + \frac{1}{q_1}\right) \left(1 + \frac{1}{q_2}\right) \cdots$$

in which the q_n are determined iteratively in the same way as described above. Here $q_{v+1} = q_v^2 - 1$ must hold, and α is rational if and only if, beginning at a certain q_n , always $q_{n+v+1} = q_{n+v}$ holds. The simple generation of product expansions which Cantor found for certain numbers like $\sqrt{2}, \sqrt{3}$, etc. is based on the fact that for

each positive number $q > 1$:

$$\sqrt{\frac{q+1}{q-1}} = \left(1 + \frac{1}{q_1}\right) \left(1 + \frac{1}{q_2}\right) \dots$$

where $q_1 = q$ and $q_{v+1} = 2q_v^2 - 1$. The ansatz

$$\frac{q+1}{q-1} = \left(1 + \frac{1}{q}\right)^2 \alpha_1$$

leads to:

$$\alpha_1 = \frac{q^2}{q^2 - 1} = \frac{2q^2 - 1 + 1}{2q^2 - 1 - 1}.$$

In the product expansion of the square root of an arbitrary rational number there will, beginning at a certain q_n , always hold $q_{n+v+1} = 2q_{n+v}^2 - 1$, but the proof for that seems not to be so easy.

Prof. Epstein (Straßburg), privy council Hensel (Marburg) and Prof. Dr. Edler (Halle) participated in the discussion.

The remark of Prof. Epstein (Straßburg) should be mentioned. He notes how, by a minor modification of the method, Cantor's product expansion as well as the expansion by unit fractions stops for the case of a "rational" number.

The speaker replied that this also follows from his expansions, but that he emphasizes to get an infinite expansion in any case.

REFERENCES

- [1] G. Fischer, <http://list.seqfan.eu/pipermail/seqfan/2018-April/018593.html>, Sequence Fanatics Discussion List, April 12, 2018.
- [2] Dr. O. Schlömilch, Dr. E. Kahl und Dr. M. Cantor, *Zwei Sätze über eine gewisse Zerlegung der Zahlen in unendliche Producte*, In: Zeitschrift für Mathematik und Physik, Verlag von B. G. Teubner, Leipzig, 1869, pp. 152–158.