

2 left to do
from Stewart

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Subject: A Handbook of Integer Sequences

Dear Dr. Sloane,

Sir, following are the various notes and additions that I have made to your handbook under the existing entries enumerated below.

Sloane's Sequence Number

- ~~8. $x^3 - y^2 = k$, the above is $|k|$.~~
- ~~15. Smallest solutions of X when $A \cdot X^2 + 1 =$ a square number.~~
- ~~30. Twice this Seq. is ~~SSN~~ 108.~~
- ~~33. primes $4k+1 = A^2 + B^2$, $A > B$ these are the Bs. see ~~SSN~~ 169.~~
- ~~36. $A_n = \text{Int}(\Phi \cdot n) - \text{Int}(\Phi \cdot (n-1))$ Pickover, Computer and the Imagination, p24.~~
- ~~68. $\dots$, 9233, 45752, 285053, 1846955, Am Sci v240n6p25.~~
- * 79a. 1, 2, 2, 2, 2, 4, 4, 2, 2, $\dots$, the nbr. of ways of expressing the positive integer $n \geq 1$ as the sum of two (ordered pair) squarefree positive integers relative prime to n . AMM v99n6p573, June/July 92.
- ~~83. of the primes. see ~~SSN~~ 85.~~
- ~~84. by primes only.~~
- ~~85. of the primes. see ~~SSN~~ 83.~~
- 89. $A_n = \text{Int}(\frac{1}{2} + \sqrt{2 \cdot n})$, Graham p97. — nice/ than what I have now
- ~~91. A Self-Generating Describing Sequence by Solomon Golomb, Concrete Math, p66.~~
- * 103a. Related to Edward Waring's problem, 1, 2, 2, 3, 4, 7, 8, 34, 32, 102, 51, 135, 150, 166, 181, $\dots$, Ian Stewart, Games, Set, and Math, p124, see ~~SSN~~ 1353.
- ~~108. half of this Sequence is ~~SSN~~ 30.~~
- ~~110. Robert Daniel Carmichael's $\lambda(n)$ function.~~
- ~~136. Companion Pell numbers, $A_0 = 2$ & $A_1 = 2$.~~

green
- new
- work
- added

7376
7377
7457
7459
2863
3626

3604
2021
6037

→ 7402

new

new
A7457

Look at
book

✓ 169. primes $4k+1 = A^2 + B^2$, $A > B$ these are the A s. see ~~SSN~~ 33.

✓ 173a. 1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 0, 1, 1, 1, 2, 1, 3, 1, 4, 1, 5, 1, 6, 1, 7, 1, 8, 1, 9, 2, 0, 2, 1, 2, 2, 2, 3, 2, 4, 2, 5, 2, 6, 2, 7, 2, 8, 2, 9, 3, 0, 3, 1, 3, 2, 3, 3, 3, 4, 3, 5, 3, 6, 3, 7, 3, 8, 3, 9, 4, ... the almost-Natural numbers, JRM v?n?p?, Math Mag v61n2p131, April 88. 7376

✓ 180. $\Phi(n)$, see ~~SSN~~ 370 & 371.

✓ 185. P^α , P is a prime and $\alpha \geq 1$.

✓ 185a. 1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 14, 15, 16, 18, 19, 24, 25, 27, 28, 31, 32, 33, 34, 35, 36, 37, 39, 49, 51, 67, 72, 76, 77, 81, 86, ... and no others < 57134 . Powers of two lacking the digit 0 in its expansion, Joseph S. Madachy, Mathematics on Vacation, p126-8. 7377

✓ 187. Joe Roberts, Lure of the Integer, p22. see ~~SSN~~ 1685. *Maybe this explains A0028*

✓ 188. Index to ~~SSN~~ 1828.

✓ 200. $\sigma(A_{n+1}) > \sigma(A_n)$.

✓ 201. Stanislaw Ulam Summation Seq., see ~~SSN~~ 231 & 909.

✓ 206. $A_1=1$, $A_{n+1} = \lfloor \sqrt{2 \cdot A_n \cdot (A_n + 1)} \rfloor = \lfloor \sqrt{2} \cdot (A_n + \frac{1}{2}) \rfloor$, Am Math Mo. v95n8p705 Oct88, & Math Mag. v64n3p168 June91.

✓ 207. $A_1=1$, $A_2=2$ & $A_3=3$, Am Math Mo. v95n8p705 Oct88.

✓ 213. see ~~SSN~~ 248.

✓ 214. $\dots, (\dots) = 2^{512} = 2^{2^9}, \dots$

✓ 223. $A_n = n + \text{Int}(\sqrt{n} + \frac{1}{2}) = n + \text{Int}(\sqrt{n} + \text{Int}(\sqrt{n}))$, Math Mag v63n1p53-5 Feb90.

✓ 228. this is Π , see ~~SSN~~ 1051.

NO 230a. 2, 3, 5, 7, 1, 1, 1, 3, 1, 7, 1, 9, 2, 3, ... an irrational decimal fraction, Am Math Mo. v100n8p779 Oct 93, HW1 5th Ed. p139. *new primes in order*

✓ 231. Stanislaw Ulam Summation Seq., see ~~SSN~~ 201 & 909.

✓ 235. also see ~~SSN~~ 982.

✓ 241. the Sieve of Eratosthenes.

✓ 242a. 1, 2, 3, 5, 7, 11, 31, 379, 1019, 1021, 2657, ..., p "Primorials" = $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot \dots \cdot p) + 1$ is prime.

✓ 246a. 2, 3, 5, 7, 11, 13, 19, 23, ..., Higg's primes in $\sum_2$, Am Math Mo v100n3p233, March 93. A7459

✓ 248. ..., 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, ... ? ..., 756839, these are the Π s of primes of the form $2^n - 1$.

✓ 256. $(\Phi^n - \hat{\Phi}^n)/\sqrt{5}$. see ~~SSN~~ 1679.

266. $A_n = 2 \cdot A_{n-1} - 1$.

A2863 322a. 1, 1, 2, 3, 7, 21, 49, 165, 552, 2176, 9988, ...; Knots, Math Mag v61n1p7 Feb 88.

326a. 1, 2, 3, 7, 23, 59, 314, 1529, 8209, ..., $A_n = (A_{n-1} \cdot A_{n-3} + A_{n-2}^2) / A_{n-4}$, for $n > 3$,
Coll. Math. Journal v24n4p393.

327. ..., 17051707, 20831323, Math of Compt v52n185p221-4 & Knuth v3p402, see ~~SSN~~ 984.

✓ 346. $\prod_{d=1}^n d_{d+n}$

358. ..., 353, 503, 613, 617, 863, by Index.

370. see ~~SSN~~ 180 & 371.

371. see ~~SSN~~ 180 & 370.

385. Ramanujan's Super Abundant $d(A_{n+1}) > d(A_n)$, Math Mag v64n5p343-6 Dec91.

391. the Lazy Caterer's Seq.

423. $A_n = A_{n-1} + A_{n-2} + A_{n-3} + A_{n-4}$.

427. $n + C(\frac{n}{4}) + C(\frac{n-1}{2})$.

429. $A_n = A_{n-1} + A_{n-2} + A_{n-3} + A_{n-4} + A_{n-5}$.

431. $A_n = A_{n-1} + A_{n-2} + A_{n-3} + A_{n-4} + A_{n-5} + A_{n-6}$.

432. $A_n = 2 \cdot A_{n-1} = \sum_{k=0}^n C(\frac{n}{k})$, Graham p231.

497. also see ~~SSN~~ 990.

509. $\text{Int}(n / \Phi^2)$ see ~~SSN~~ 917.

511. Leonard Euler's Gen. Pentagonal Nbrs. $\frac{n}{2} \cdot (3 \cdot n - 1)$, $n \geq \pm 1$.

522. $= C(\frac{n}{2}) - n$, the nbr of diagonals of a regular convex n-agon, Math. Mag v61n1p28.

529. $X^2 - D \cdot Y^2 = -1$, SI1 2nd p334.

531a. 1, 2, 5, 10, 17, 28, ..., Surds of period 1, $= n^2 + 1$.

552. $[(1+\sqrt{2})^n - (1-\sqrt{2})^n] / \sqrt{8}$, Knuth v2p605, David Wells p90 & SI1 p327.

561. Whys & Wherefores p92. Ball&Coxeter p112. Computers in Nbr. Theory p363.

569. JRM v13n2p158.

$$577. A_n = \frac{(2 \cdot n)!}{n!(n+1)!} = \frac{1}{n+1} \cdot C\left(\frac{2 \cdot n}{n}\right) = \frac{2}{n+1} \cdot C\left(\frac{2n+1}{n}\right) = \frac{1}{n+1!} \cdot 2 \cdot 6 \cdot 10 \cdots (4 \cdot n - 10) = [A_{n-1} \cdot (4 \cdot n - 6)]/n,$$

$$B_{n+1} = B_0 \cdot B_n + B_1 \cdot B_{n-1} + B_2 \cdot B_{n-2} + \cdots + B_{n-1} \cdot B_1 + B_n \cdot B_0.$$

Math Mag v61n4p211 Oct88.

$$589. A_0 = 0, A_{n+1} = n \cdot A_n + 1.$$

603. Convergents to Lehmers Constant $\xi - 1$, see ~~SSN~~ 1230.

617. $\cdots$, 272, 306, Kissing nrs. David Wells p84, also called Newton, Contact, Coordination and Ligancy numbers.

$$625. A_n = 2 \cdot A_{n-1} + 2.$$

651. $\cdots$, 275750636070, 1633292229030, 9737153323590, $\cdots$.

668. see ~~SSN~~ 1081.

$$695. \text{Int}(e^n).$$

731. Ball&Coxeter p112.

$$742. 2^n n!.$$

$$766. \ln n = P!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \pm \frac{1}{P!}), \text{ David Wells p122.}$$

$$773. \prod_{k=0}^n C\left(\frac{n}{k}\right), \text{ Hyperfactorial / Superfactorial, } \del{SSN} 1514/\del{SSN} 811, \text{ Graham p231.}$$

775. Kraitchik p250.

786a. 2, 11, 13, 17, $\cdots$ they nor their square can be expressed as $a^2 + 5 \cdot b^2$, Beiler p285, see ~~SSN~~ 2264a.

787. Am Math Mag v97n7p625.

$$789. S\left[\frac{n+2}{n}\right].$$

$$811. \prod_{k=1}^n k!, \text{ Superfactorial, Graham p231.}$$

891. see ~~SSN~~ 1336.

909. $\cdots$, 176, 187, 192, 196, $\cdots$, Stanislaw Ulam Summation Seq., see ~~SSN~~ 201 & 231.

911. $\cdots$, 571, $\cdots$? $\cdots$, 2971, 4723, 5387, by Index, Pickover Mazes p350.

$$917. A_n = \text{Int}(n/\Phi) = \text{Int}(n \cdot \Phi), \text{ see } \del{SSN} 509.$$

$$921. \sigma(n).$$

A3626

924. $A_0 = 2$ & $A_1 = 1$, $\Phi^n + \hat{\Phi}^n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$, see ~~SSN~~ 1679.

941. Polyiamond.

973a. ~~1, 1, 1, 1, 1, 1, 3, 5, 9, 23, 75, 421, 1103, 5047, 41783, 281527, 2534423, 14161887,~~
~~232663909, 3277905290, ..., $A_n = (A_{n-1} * A_{n-5} + A_{n-2} * A_{n-4} + A_{n-3}^2) / A_{n-6}$,~~
~~Somos, Mathematical Intelligencer v13n1p40, Pickover, Mazes p351.~~

982. see ~~SSN~~ 235.

984. Math of Compt. v52n185p221-4, see ~~SSN~~ 327.

988. Pascal's Triangle (mod 2) read in decimal, Math Mag v63n1p3 Feb90.

990. see ~~SSN~~ 497.

1002. Sum of the Integers, $C(\frac{n}{2})$, $A_{n+1} = [n \cdot A_n + 2 \cdot (n+1)^2] / (n+2)$.

1035. Sieve of Stanislaw Ulam.

1039. Gaussian Primes of real nbrs.

1048. Josephus Flavius sieve.

1051. $= n^2 + n + 1$, see ~~SSN~~ 228.

1059. Stirling Nrbs. of the 2nd Kind $S\{\frac{n}{2}\}$, $A_n = 2 \cdot A_{n-1} + 1$.

1064. $\frac{1}{2} [(1+\sqrt{2})^n + (1-\sqrt{2})^n]$.

1070. ..., 377379369, 1105350729, ..., $\sum_{j=0}^{n/3} (-1)^j \cdot C(\frac{n}{j}) \cdot C(\frac{2 \cdot n - 1 - 3 \cdot j}{n-1})$.

1071. $\text{Int}(e^n + \frac{1}{2})$.

1072. Polyhexes.

1072a. 3, 7, 23, ..., primes which cannot be expressed in the form $a^2 + 5 \cdot b^2$ although their squares can. Beiler p284, see ~~SSN~~ 2264a.

1081. ~~SSN~~ 688 (the "Primorials") $+1 = (2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot \dots \cdot p) + 1$, p is prime, AmMathMo v95n8p699, see ~~SSN~~ 242a.

1084. "decimated" Lucas Seq., Number Theory, Springer-Verlag v7p73 & SI1 2nd Ed.

1087. the periodicities of the Fibonacci sequence modulus n , Lure of the Integers p162.

1089. N.L. Gilbreath hypothesis 1958, conjecture, $A_{n+1} > A_n - 2$.

1101. $\dots, 1836311903, \dots, = \text{Int}\left[\frac{1}{\sqrt{5}} \cdot \left(\frac{3+\sqrt{5}}{2}\right)^n\right]$. Niven & Zuckerman, 4th Ed. p123.

$$1130. = \frac{3}{n+1} \cdot C\left(\frac{2n-1}{n}\right).$$

1132a. $\pi(n!)$ for $n > 1$, = 1, 3, 9, 30, 128, 675, 4231, 30969, 258689, 2428956, 25306287, 289620751, 3610490805, $\dots$. 3604

$$1142. = \text{Int}(\pi^{n+\frac{1}{2}}).$$

1163. $n \cdot A_n = 3 \cdot (2n-3) \cdot A_{n-1} - (n-3) \cdot A_{n-2}$, Vardi p198.

$$1165. S\left[\frac{n}{2}\right], A_{n+1} = n \cdot A_n + (n-1)!.$$

1183. Cyclic numbers, SPR p146.

1215. $\dots, 5187, 10604, 11714, 13365, 18315, 22935, 25545, 32864, 38804, 39524, 46215, 48704, \dots$, $\Phi(n) = \Phi(n+1)$, see ~~SSN~~ 1328.

$$1217. \prod_{n=1}^{\frac{n}{2}} 2 \cdot n - 1, n! / (2^{\frac{n}{2}} * \frac{n}{2}!) \text{ for even } n$$

1230. see ~~SSN~~ 603.

$$1231. \frac{1}{2} [(1+\sqrt{2})^{2 \cdot n} + (1-\sqrt{2})^{2 \cdot n}].$$

1233. Graham, Knuth & Patashnik, Concrete Mathematics, p528.

1247. $A_n = 6 \cdot A_{n-1} - A_{n-2} + 2 = \frac{1}{4} [(\sqrt{2}+1)^{2 \cdot n+1} - (\sqrt{2}-1)^{2 \cdot n+1} - 2]$, with Consecutive Legs (lesser given), Beiler p123, see ~~SSN~~ 1630.

1262. 285311670601, 98077104930805, 302875106592241, 144456088732254195, $\dots$, for even n , $X = n^{n+1} - n^n - n + 1$, for odd n , $X = n^n - n + 1$. A2021

1309. Murphy sequence, JourRecMath v10n3p230.

1313. Martin Gardner, Penrose Tiles To Trapdoor Ciphers, p69.

1322. 65, 66, $\dots$, eg. not the primes ~~SSN~~ 241.

1323. Semi-Primes.

$$1327. = \text{Int}\left[\frac{1}{2} (7 + \sqrt{48 \cdot n + 1})\right], \text{ Am.Math.Mo. v98n9p874.}$$

1328. $\dots, 1082, 1226, 1322, 1330, 1346, 1466, 1514, 1608, 1754, 1994, 2132, \dots$, $\Phi(n) = \Phi(n+2)$, see ~~SSN~~ 1215.

1336. N^α , $\alpha > 1$, see ~~SSN~~ 891.

1352a. Related to Edward Waring's problem, 1, 4, 9, 19, 21, 31, 45, 62, 82, 102, 120, 135, 150, 166, 181, ..., Ian Stewart, Games, Set, and Math, p123, see ~~SSN~~ 1353.

1353. $\sim \text{Int}[(\frac{3}{2})^k] + 2^k - 2$, Lure of the Integers, p138.

1363. the Sum of the Triangular Nbrs. $\frac{n}{6} \cdot (n+1) \cdot (n+2) = \sum_{k=1}^n \frac{n \cdot (n+1)}{2}$.

1385. ..., 740461601, 2012783315, 5471312310, 1487256883, ..., when k exceeds n of $\sum_{n=1}^k \frac{1}{n}$
 Jour Rec Math v10n2p124 1977 & Math Mag v65n5p308 Dec92.

1415. ..., 1188640, 4345965, ..., $= \frac{4}{n+1} \cdot C(\frac{2n-3}{n})$.

1463. ..., 92897280, 1857945600, 40874803200, ..., $A_1 = 1, A_{n+1} = 2 \cdot n \cdot A_n = 2^{n-1} \cdot n!$.

1514. Hyperfactorial, Graham, Knuth & Patashnik, Concrete Mathematics, p477.

1524. Wilson Quotients, not remainders.

1530a. Congruent numbers,

1558. $= n^2 - n - 1$, no index listing.

1562a. $n > 0$.

1569. ..., 411, 525, ..., $\frac{1}{16} \cdot [2 \cdot n \cdot (n+2) \cdot (2 \cdot n+1) + (-1)^n - 1]$, $T_0 = 0$ & $T_1 = 1$, $T_{n+1} = T_n + T_{n-1} - T_{n-2} + 3 \cdot n + 1$, Coll Math Journ v20n5p370-84, Math Mag v66n1p40.

1574. the Sum of the Integers Squared, $= \frac{n}{6} \cdot (n+1) \cdot (2n+1) = \sum_{k=1}^n n^2$.

1578. $= \frac{n}{4!} \cdot (n+1) \cdot (n+2) \cdot (n+3)$.

1602. $\frac{5}{n+1} \cdot C(\frac{2n-4}{n})$.

1608. ..., 5461, 21845, 87381, 349525, 1398101, 5592405, $A_1 = 1, A_n = 4 \cdot A_{n-1} + 1 = \frac{1}{3}(4^n - 1)$.

1630. $A_n = 6 \cdot A_{n-1} - A_{n-2} = \frac{1}{\sqrt{8}} [(\sqrt{2} + 1)^{2 \cdot n+1} + (\sqrt{2} - 1)^{2 \cdot n+1}]$.

with Consecutive Legs (Hypotenuses given), Beiler p123, see ~~SSN~~ 1247,

1636. $(A^5 - B^5)/D$, $A > B \geq D \geq 1$.

1678. the least ε such that $10^\varepsilon \equiv 1 \pmod n$, $n \nmid 10$ & $n > 6$, see ~~SSN~~ 1680.

1679. $\Phi = \frac{1}{2}[\sqrt{5} + 1]$ & $\hat{\Phi} = \frac{1}{2}[\sqrt{5} - 1]$.

1680. see ~~SSN~~ 1678.

1685. Joe Roberts, Lure of the Integers, p22. see ~~SSN~~ 187.

1714. $\frac{n}{12} \cdot (n+1)^2 \cdot (n+2)$

1718. $C\left(\frac{C(\frac{n}{2})+1}{2}\right)$

1719. $\frac{n}{5!} \cdot (n+1) \cdot (n+2) \cdot (n+3) \cdot (n+4)$.

1734. $S\{\frac{n}{3}\}$.

1760. Triangular numbers which are squares. The above numbers are the square roots.

1762. $S[\frac{n}{3}]$.

~~1768. QUESTION. Should not the eighth entry be 30720 instead of 30270? $2^x * (2 \cdot n + 1)$.~~

1779. $S[\frac{n+3}{n}]$.

1783. Joe Roberts, Lure of the Integers, p208.

1799. $A_n = \prod_1^x P_x$, $x = 2, 4, 9, 22, \dots$, ?

1819. $\equiv 6 \cdot n + 1$, no primes of the form $3 \cdot n - 1$.

1823. Cyclic numbers of 1, SPR p143.

1828. Prime Hex. nbrs., Hexagonal clusters - "Polyhedral Dissections."

$$A_n = 1 + 6 + 12 + \dots + 6 \cdot n = n^3 - (n-1)^3 = 3 \cdot n^2 + 3 \cdot n + 1 = 1 + \sum_{k=1}^n 6 \cdot k.$$

1834. $\frac{1}{2} \{[(1+\sqrt{6})^n + (1-\sqrt{6})^n]\}$.

1845. $S\{\frac{n+2}{n}\} = \frac{n}{4!} \cdot (n+1) \cdot (n+2) \cdot (3 \cdot n + 1)$.

1847. $= \frac{n}{6!} \cdot (n+1) \cdot (n+2) \cdot (n+3) \cdot (n+4) \cdot (n+5)$.

1866. $\frac{7}{n+1} \cdot C(\frac{2n-6}{n})$.

1869. $\frac{1}{2} \cdot [(1 + \sqrt{2})^{2 \cdot n + 1} + (1 - \sqrt{2})^{2 \cdot n + 1}]$.

1886. Denominator of successive continued fractions of $\overline{\pi}$ "Pi bar" = $\pi - 3$,

1905. n^3 .

1907. $\frac{1}{6} \cdot n^2 \cdot (n+1) \cdot (n+2)$.

might explain this one

1924. the above are the indexes, $T_n = \frac{1}{2} \cdot n \cdot (n+1)$, see ~~SSN~~ 1760.

1935. Egyptian Fraction's Denominators. ..., 162924332716605980,

1970. $\frac{n}{4!} \cdot (n+1) \cdot (n+2) \cdot (5 \cdot n - 1)$.

1981. $\frac{9}{n+1} \cdot C(\frac{2 \cdot n - 8}{n})$.

2018. $S\{\frac{n}{4}\}$.

2022. $S[\frac{n}{4}]$.

2044. $(A^5 + B^5)/D$, $A > B \geq 1$, $26 \geq D \geq 1$.

2048. $\frac{11}{n+1} \cdot C(\frac{2 \cdot n - 10}{n})$.

2091. the denominators of James Stirling's formula

2100. $(A^6 + B^6)/D$, $A > B \geq 1$, $240 \geq D \geq 1$.

2104. $\frac{13}{n+1} \cdot C(\frac{2 \cdot n - 12}{n})$.

2105. Semi-Cuban Primes, $\frac{1}{2}[n^3 - (n-2)^3]$.

2120. 14^n .

2136. $S\{\frac{n+2}{n}\}$.

2141. $S\{\frac{n}{5}\}$.

2142. $S[\frac{n}{5}]$.

2178. ..., 65617, 66161, ..., $A^4 + B^4$, $A > B \geq 1$.

2215. $S\{\frac{n}{6}\}$.

2216. $S[\frac{n}{6}]$.

2239. $S[\frac{n+4}{n}]$.

2262. $\sum_{n=0} (2 \cdot n + 1)^3$.

2263. $S\{\frac{n}{7}\}$.

2264. $S[\frac{n}{7}]$.

2264a. 29, 41, 61, ..., primes which can be expressed as $A^2 + 5 \cdot B^2$, Beiler p284,
see ~~SSN~~ 786a & 1072a.

2272. $S\{\frac{n+4}{n}\}$.

2277. n^5 .

2292. SPR 98, & Stewart p29.

2293a. Surds of a periodicities of 3: 41, 130, 269, 370, 458, 697, 986, ..., $= (5 \cdot k + 1)^2 + 4 \cdot k + 1$, SHI p323-4.

2294. $\frac{1}{2}(A^4 + B^4)$, $A > B \geq 1$.

2311. SciAm. v242n6p30.

2323a. Primitive Weird numbers: 70, 836, 4030, 5830, 7192, 7912, 9272, 10792, 17272, 45356, 73616, 83312, 91388, 113072, 243892, 254012, 338572, 343876, 388076, 519712, 539744, 555616, 682592, 786208, 1188256, 1229152, 1713592, 1901728, 2081824, 2189024, 3963968, 4128448, 4145216, 4199030, 4486208, 4559552, 4632896, 4790812, 4960448, 5440192, 5568448, 6460864, 6621632, 7354304, 7470272, 8000704, 8134208, 8812312, 9928792, 11339816, 11547352, ..., Stephen P. Richard p85.

2333a. tri-Perfect numbers: 120, 672, 523776, 459818240, 1476304896, 31001180160, ..., David Wells p135.

2347a. Prime Cullen numbers by Index $C_n = n \cdot 2^n + 1$, for n s: 1, 141, 4713, 5795, 6611, 18496, and no others ≤ 20000 . Paulo Ribenboim, p175.

2352. see ~~SSN~~ 2363.

2363. see ~~SSN~~ 2352.

2365. base Two - PseudoPrimes, $2^{p-1} \equiv 1 \pmod{p}$.

2372a. the number of Magic Squares of order n : 1, 0, 1, 880, 275305224, ..., David Wells p190.

additional Bibliography

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Lucky

If at some future date, I run across an addition, I will forward the same to you.

Sequentially yours,



Robert G. Wilson v
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A CURIOUS BINOMIAL IDENTITY

NEIL J. CALKIN

In this note we shall prove the following curious identity of sums of powers of the partial sum of binomial coefficients.

1. AN IDENTITY

Theorem. $\sum_{l=0}^n \left(\sum_{k=0}^l \binom{n}{k} \right)^3 = n 2^{3n-1} + 2^{3n} - \frac{3n}{4} 2^n \binom{2n}{n}.$

Proof. Define $f_n = \sum_{l=0}^n \left(\sum_{k=0}^l \binom{n}{k} \right)^3$. It is sufficient to show that

$$f_{n+1} - 8f_n = 4 \cdot 2^{3n} - 3 \cdot 2^n \binom{2n}{n}$$

Write $A_l = \sum_{k=0}^l \binom{n}{k}$. Then $f_n = \sum_{l=0}^n A_l^3$.

$$\begin{aligned} f_{n+1} &= \sum_{l=0}^{n+1} \left(\sum_{k=0}^l \binom{n+1}{k} \right)^3 \\ &= 2^{3n+3} + \sum_{l=0}^n \left(\sum_{k=0}^l \binom{n+1}{k} \right)^3 \\ &= 2^{3n+3} + \sum_{l=0}^n \left(\sum_{k=0}^l \binom{n}{k} + \binom{n}{k-1} \right)^3 \\ &= 2^{3n+3} + \sum_{l=0}^n \left(2A_l - \binom{n}{l} \right)^3 \\ f_{n+1} - 8f_n &= 2^{3n+3} + \sum_{l=0}^n \left(2A_l - \binom{n}{l} \right)^3 - (2A_l)^3 \\ &= 2^{3n+3} - \sum_{l=0}^n 12A_l^2 \binom{n}{l} + \sum_{l=0}^n 6A_l \binom{n}{l}^2 - \sum_{l=0}^n \binom{n}{l}^3 \end{aligned}$$

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Observation 1:

$$\sum_{l=0}^n A_l \binom{n}{l}^2 = \frac{1}{2} 2^n \binom{2n}{n} + \frac{1}{2} \sum_{l=0}^n \binom{n}{l}^3$$

Indeed;

$$\begin{aligned} \sum_{l=0}^n A_l \binom{n}{l}^2 &= \sum_{l=0}^n A_{n-l} \binom{n}{n-l}^2 \\ &= \sum_{l=0}^n A_{n-l} \binom{n}{l}^2 \end{aligned}$$

and since

$$A_l + A_{n-l} = 2^n + \binom{n}{l}$$

we have

$$\begin{aligned} \sum_{l=0}^n A_l \binom{n}{l}^2 &= \frac{1}{2} \sum_{l=0}^n \left(2^n + \binom{n}{l} \right) \binom{n}{l}^2 \\ &= \frac{1}{2} \sum_{l=0}^n 2^n \binom{n}{l}^2 + \frac{1}{2} \sum_{l=0}^n \binom{n}{l}^3 \\ &= \frac{1}{2} 2^n \binom{2n}{n} + \frac{1}{2} \sum_{l=0}^n \binom{n}{l}^3 \end{aligned}$$

Observation 2:

$$\sum_{l=0}^n A_l^2 \binom{n}{l} = \frac{2^{3n}}{3} + \frac{1}{2} 2^n \binom{2n}{n} + \frac{1}{6} \sum_{l=0}^n \binom{n}{l}^3$$

Indeed,

$$\begin{aligned}
 2^{3n} &= A_n^3 = \sum_{l=0}^n A_l^3 - A_{l-1}^3 \\
 &= \sum_{l=0}^n A_l^3 - \left(A_l - \binom{n}{l} \right)^3 \\
 &= \sum_{l=0}^n 3A_l^2 \binom{n}{l} - \sum_{l=0}^n 3A_l \binom{n}{l}^2 + \sum_{l=0}^n \binom{n}{l}^3 \\
 &= \sum_{l=0}^n 3A_l^2 \binom{n}{l} - \frac{3}{2} 2^n \binom{2n}{n} - \frac{1}{2} \sum_{l=0}^n \binom{n}{l}^3
 \end{aligned}$$

Hence

$$\sum_{l=0}^n A_l^2 \binom{n}{l} = \frac{2^{3n}}{3} + \frac{1}{2} 2^n \binom{2n}{n} + \frac{1}{6} \sum_{l=0}^n \binom{n}{l}^3$$

Putting these together, we indeed find that

$$f_{n+1} - 8f_n = 4 \cdot 2^{3n} - 3 \cdot 2^n \binom{2n}{n}$$

as required. $\square$

2. AN APPLICATION

In this section we shall discuss an application of this to order statistics. Observe that the expected value of the maximum of three independent Bernoulli random variables $B(n, \frac{1}{2})$ is

$$\begin{aligned}
 \sum_{l=0}^n \left(1 - \left(\sum_{k=0}^l 2^{-n} \binom{n}{k} \right)^3 \right) &= n - 2^{-3n} f_n \\
 &= \frac{n}{2} + \frac{3}{4} n 2^{-2n} \binom{2n}{n}.
 \end{aligned}$$

Hence, by the central limit theorem, the expected value m_3 of the maximum of three independent normal $N(0, 1)$ random variables is

$$m_3 = \lim_{n \rightarrow \infty} \frac{\frac{3}{4} n 2^{-2n} \binom{2n}{n}}{\frac{\sqrt{n}}{2}} = \frac{3}{2\sqrt{\pi}}$$

subtracting off the mean, dividing by the standard deviation and applying Stirling's formula for the asymptotics of $n!$

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May 12, 1994

Neil J. Calkin
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Dear Dr. Calkin,

I have just seen your "Curious Binomial Identity", and observe that another derivation is possible. As you remark, we have

$$2^{-3n} f_n = n - E(\max(X_1, X_2, X_3))$$

where the X 's are independent Bernoulli $(n, 1/2)$. So by symmetry the result is equivalent to

$$E(\max(X_1, X_2, X_3) - \min(X_1, X_2, X_3)) = \frac{3n}{2} \frac{1}{2^{2n}} \binom{2n}{n}$$

But

$$\max(x, y, z) - \min(x, y, z) = \frac{1}{2} (\max(x, y) - \min(x, y) + \max(x, z) - \min(x, z) + \max(y, z) - \min(y, z))$$

so it is sufficient to show

$$E(\max(X_1, X_2) - \min(X_1, X_2)) = \frac{n}{2^{2n}} \binom{2n}{n} \quad (1)$$

Here is a cute way to do this. The expectation is

$$\begin{aligned} & \sum_{k=0}^n \sum_{l=0}^n \frac{1}{2^{2n}} \binom{n}{k} \binom{n}{l} |k-l| \\ &= \sum_{k=0}^n \sum_{l=0}^n \frac{1}{2^{2n}} \binom{n}{k} \binom{n}{l} \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - e^{2ix(k-l)}) \frac{dx}{x^2} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(1 - \left(\frac{1+e^{2ix}}{2}\right)^n \left(\frac{1+e^{-2ix}}{2}\right)^n\right) \frac{dx}{x^2} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - \cos^{2n} x) \frac{dx}{x^2} = I_n \quad \text{say} \end{aligned}$$

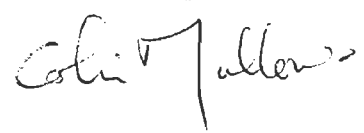
To evaluate this, consider the generating function

$$\begin{aligned} \sum_{n=1}^{\infty} I_n y^{n-1} &= \frac{1}{2\pi} \frac{1}{1-y} \int_{-\infty}^{\infty} \frac{1 - \cos^2 x}{1 - y \cos^2 x} \frac{dx}{x^2} \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{k\pi}^{k\pi+\pi} = \frac{1}{2\pi} \frac{1}{1-y} \int_0^{\pi} \frac{1 - \cos^2 x}{(1 - y \cos^2 x)} \sum_{k=-\infty}^{\infty} \frac{1}{(x+k\pi)^2} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2\pi} \frac{1}{1-y} \int_0^\pi \frac{1}{1-y\cos^2 x} dx \\ &= \frac{1}{2} (1-y)^{-3/2} = \frac{d}{dy} (1-y)^{-1/2} = \frac{d}{dy} \sum_{n=0}^{\infty} \frac{y^n}{2^{2n}} \binom{2n}{n} \end{aligned}$$

and this proves (1).

Sincerely,



Colin L. Mallows