Monoids of Natural Numbers

Steven Finch

March 17, 2009

Let $\mathbb{N}$ denote the set of nonnegative integers. If $A = \{a_1, a_2, \ldots, a_m\}$ is a set of positive integers satisfying $\gcd(a_1, a_2, \ldots, a_m) = 1$, then

$$\langle a_1, a_2, \ldots, a_m \rangle = \left\{ \sum_{j=1}^{m} x_j a_j : x_j \in \mathbb{N} \text{ for each } 1 \le j \le m \right\}$$

is the subset of $\mathbb{N}$ **generated by** A. For example,

$$\langle a, a+1, a+2, a+3, \ldots, 2a-1 \rangle = \{0\} \cup \{a, a+1, a+2, a+3, \ldots\}$$

and

$$\langle 2, b \rangle = \{0, 2, 4, \ldots, b-3\} \cup \{b-1, b, b+1, b+2, b+3, \ldots\}$$

when $b \ge 3$ is odd.

A **numerical monoid** S is a subset of $\mathbb{N}$ that is closed under addition, contains 0, and has finite complement in $\mathbb{N}$. (Most authors use the phrase "numerical semigroup", but semigroups by definition need not contain 0, hence the usage is puzzling.) The **Frobenius number** f of S is the maximum element in the set $\mathbb{N} - S$, and the **genus** g of S is the cardinality of $\mathbb{N} - S$. Therefore

$$f(\langle a, a+1, a+2, a+3, \ldots, 2a-1 \rangle) = a-1, \qquad f(\langle 2, b \rangle) = b-2,$$

$$g(\langle a, a+1, a+2, a+3, \ldots, 2a-1 \rangle) = a-1, \qquad g(\langle 2, b \rangle) = (b-1)/2$$

and, more generally [1],

$$f(\langle a, b \rangle) = (a-1)(b-1) - 1, \qquad g(\langle a, b \rangle) = (a-1)(b-1)/2$$

when $\gcd(a, b) = 1$. It is known that $f + 1 \le 2g$ always [2, 3]. Table 1 gives all monoids S with $1 \le f \le 4$ or $1 \le g \le 4$.

Table 1. *Numerical Monoids with Small Frobenius Number or Genus*

$f=1$	$f=2$	$f=3$	$f=4$	$g=1$	$g=2$	$g=3$	$g=4$
$\langle 2,3\rangle$	$\langle 3,4,5\rangle$	$\langle 4,5,6,7\rangle$	$\langle 5,6,7,8,9\rangle$	$\langle 2,3\rangle$	$\langle 3,4,5\rangle$	$\langle 4,5,6,7\rangle$	$\langle 5,6,7,8,9\rangle$
		$\langle 2,5\rangle$	$\langle 3,5,7\rangle$		$\langle 2,5\rangle$	$\langle 3,5,7\rangle$	$\langle 4,6,7,9\rangle$
						$\langle 3,4\rangle$	$\langle 3,7,8\rangle$
						$\langle 2,7\rangle$	$\langle 4,5,7\rangle$
							$\langle 4,5,6\rangle$
							$\langle 3,5\rangle$
							$\langle 2,9\rangle$

Define sequences [4, 5, 6, 7]

$$\{F_n\}_{n=1}^{\infty} = \{1,1,2,2,5,4,11,10,\ldots\},$$

$$\{G_n\}_{n=1}^{\infty} = \{1,2,4,7,12,23,39,67,\ldots\}$$

by

$$F_n = (\text{the number of monoids } S \text{ with } f(S)=n)\,,$$

$$G_n = (\text{the number of monoids } S \text{ with } g(S)=n)$$

then Backelin [8] showed that

$$0 < \liminf_{n\to\infty} 2^{-n/2}F_n < \limsup_{n\to\infty} 2^{-n/2}F_n < \infty,$$

$$\tfrac{1}{2}(2.47) < \lim_{\substack{n\to\infty \\ n\equiv 0 \bmod 2}} 2^{-n/2}F_n < \tfrac{1}{2}(3.3), \qquad \tfrac{1}{\sqrt{2}}(2.5) < \lim_{\substack{n\to\infty \\ n\equiv 1 \bmod 2}} 2^{-n/2}F_n < \tfrac{1}{\sqrt{2}}(3.32)$$

and Bras-Amorós [5, 9, 10] conjectured that

$$\lim_{n\to\infty}\frac{G_{n+1}}{G_n} = \varphi$$

where $\varphi = (1+\sqrt{5})/2 = 1.6180339887...$ is the Golden mean. Tighter bounds are needed for F_n asymptotics; it has not even been proved that G_n is increasing.

A monoid is **irreducible** if it cannot be written as the intersection of two monoids properly containing it [11]. A monoid S is irreducible if and only if S is maximal (with respect to set inclusion) in the collection of all monoids with Frobenius number $f(S)$. Irreducible monoids with odd f are the same as **symmetric** monoids (for which $f = 2g-1$ always); irreducible monoids with even f are the same as **pseudo-symmetric** monoids (for which $f = 2(g-1)$ always). As an example, $\langle 3,4\rangle$ and $\langle 2,7\rangle$ are the two symmetric monoids with Frobenius number 5; $\langle 4,5,7\rangle$ is the unique pseudo-symmetric monoid with Frobenius number 6. Another characterization of symmetry and pseudo-symmetry will be given shortly. Define [4, 12]

$$\{H_n\}_{n=1}^{\infty} = \{1, 1, 1, 1, 2, 1, 3, 2, 3, 3, 6, 2, 8, \ldots\}$$

by

$$H_n = (\text{the number of irreducible monoids } S \text{ with } f(S) = n)$$

then Backelin [8] showed that

$$0 < \liminf_{n\to\infty} 2^{-n/6} H_n < \limsup_{n\to\infty} 2^{-n/6} H_n < \infty,$$

$$\tfrac{1}{2}(9.36) < \lim_{\substack{n\to\infty \\ n\equiv 0 \bmod 6}} 2^{-n/6} H_n = \tfrac{1}{\sqrt{2}} \lim_{\substack{n\to\infty \\ n\equiv 3 \bmod 6}} 2^{-n/6} H_n < c.$$

No finite value c (as an upper bound for H_n asymptotics) has been rigorously proved.

0.1. Sets without Closure. A **numerical set** S is a subset of $\mathbb{N}$ that contains 0 and has finite complement in $\mathbb{N}$. The **Frobenius number** of S is, as before, the maximum element in the set $\mathbb{N} - S$. Nothing has been assumed about additivity so far. Every numerical set S has an associated **atom monoid** $A(S)$ defined by

$$A(S) = \{n \in \mathbb{Z} : n + S \subseteq S\}.$$

Clearly $A(S) \subseteq S$; also $A(S) = S$ if and only if S is itself a numerical monoid. The Frobenius number of $A(S)$ is the same as the Frobenius number of S; thus there is no possible ambiguity when speaking about $f(S)$. Let

$$\mathbb{N}_n = \langle n+1, n+2, n+3, \ldots, 2n+1\rangle = \{0\} \cup \{n+1, n+2, n+3, \ldots\}$$

which we already know has Frobenius number n. Given n, which sets S have $A(S) = \mathbb{N}_n$? Table 2 answers the question for $1 \le n \le 5$. For brevity, we give only T, where $S = T \cup \mathbb{N}_n$ is a disjoint union.

Table 2. *Numerical Sets $T \cup \mathbb{N}_n$ with Atom Monoid $\mathbb{N}_n$*

$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$\emptyset^*$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
	$\{1\}$	$\{1\}^*$	$\{1\}$	$\{1\}$
		$\{1,2\}$	$\{2\}$	$\{2\}$
			$\{1,2\}$	$\{1,2\}^*$
			$\{1,3\}$	$\{1,3\}^*$
			$\{1,2,3\}$	$\{1,4\}$
				$\{2,3\}$
				$\{1,2,3\}$
				$\{1,2,4\}$
				$\{1,2,3,4\}$

Define [13]

$$\{P_n\}_{n=1}^{\infty} = \{1, 2, 3, 6, 10, 20, 37, 74, \ldots\}$$

by

$$P_n = (\text{the number of sets } S \text{ with } A(S) = \mathbb{N}_n)$$

then Marzuola & Miller [14] showed that

$$\lim_{n\to\infty} \frac{P_n}{2^{n-1}} \approx 0.484451 \pm 0.005.$$

Also, a numerical set S with Frobenius number n satisfying

$$x \in S \text{ if and only if } n - x \notin S$$

is **symmetric** if n is odd and **pseudo-symmetric** if n is even and $n/2 \notin S$ (we agree to exclude $x = n/2$ from consideration). The symmetric cases in Table 2 are marked by $*$. Define [13]

$$\{Q_k\}_{k=1}^{\infty} = \{1, 1, 2, 3, 6, 10, 20, 37, 73, \ldots\}$$

by

$$Q_k = (\text{the number of symmetric sets } S \text{ with } A(S) = \mathbb{N}_{2k-1})$$

then [14]

$$\lim_{k\to\infty} \frac{Q_k}{2^{k-1}} \approx 0.230653 \pm 0.006.$$

It is interesting the Q_{k+2} is the number of additive 2-bases for $\{0, 1, 2, \ldots, k\}$, meaning sets Σ that satisfy

$$\Sigma \subseteq \{0, 1, 2, \ldots, k\} \subseteq \Sigma + \Sigma.$$

The asymptotics for the corresponding "anti-atom" problem for pseudo-symmetric sets are identical to the preceding.

Addendum. , Work continued on the growth of $G(n)$ [15, 16], culminating with a theorem by Zhai [17]:

$$\lim_{n\to\infty} \frac{G(n)}{\varphi^n} \qquad \text{exists and is finite (and is at least 3.78).}$$

No similar progress can be reported for $F(n)$.

A **more sums than differences** (MSTD) set is a finite subset S of $\mathbb{N}$ satisfying $|S + S| > |S - S|$. The probability that a uniform random subset of $\{0, 1, ..., n\}$ is an MSTD set is provably > 0.000428 and conjecturally ≈ 0.00045, as $n \to \infty$.

Underlying solution techniques [18] resemble those in [16]; the problem itself reminds us of [19].

Given $\gcd(a,b,c)=1$, let $\tilde{f}(a,b,c)=f(\langle a,b,c\rangle)+a+b+c$. Ustinov [20, 21] proved that, on average, $\tilde{f}(a,b,c)$ is asymptotic to $(8/\pi)\sqrt{abc}$. The following probability density function

$$p(t)=\begin{cases}\dfrac{12}{\pi}\left(\dfrac{t}{\sqrt{3}}-\sqrt{4-t^2}\right) & \text{for } \sqrt{3}\le t\le 2\\ \dfrac{12}{\pi^2}\left[\sqrt{3}\,t\arccos\left(\dfrac{t+3\sqrt{t^2-4}}{4\sqrt{t^2-3}}\right)+\dfrac{3}{2}\sqrt{t^2-4}\ln\left(\dfrac{t^2-4}{t^2-3}\right)\right] & \text{for } t>2\end{cases}$$

describes more fully the behavior of $\tilde{f}(a,b,c)/\sqrt{abc}$ as $\max\{a,b,c\}\to\infty$; in particular, the distribution has a sharp peak at mode 2 and has mean

$$\int\limits_{\sqrt{3}}^{\infty} t\,p(t)dt=\frac{8}{\pi}.$$

In words, $\tilde{f}(a,b,c)$ is the largest positive integer not representable as $xa+yb+zc$ for *positive* coefficients x, y, z. This is more convenient for the analysis – based on continued fractions (Porter's constant [22] appears in [20]) – leading to proof of such limiting results. Let $\tilde{g}(a,b,c)$ denote the cardinality of all positive integers not representable as $xa+yb+zc$, $x>0$, $y>0$, $z>0$. One of Ustinov's students calculated the average normalized genus to be $8/\pi-64/(5\pi^2)$; we await the proof. Also, what can be said about rates of growth of $F_{n,k}$ and $G_{n,k}$, the counts of monoids when the number of generators is fixed to be k?

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