

The triangle for the Binomial Coefficients is the father of a family of numerical triangles. Jack Ramsay, consultant in the Department of Radar Systems, shows how some of these triangles have application to the theory of array antennas.

ON ARITHMETICAL TRIANGLES

Blaise Pascal (1623-1662), French scientist and writer on religion, is credited with the construction of the first digital calculating machine. He is better known, however, for his famous "Arithmetical Triangle" for the Binomial Coefficients published posthumously in 1665, 300 years ago, and shown below:

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		1			
	1	2	1		
	1	3	3	1	
1	4	6	4	1	
1	5	10	10	5	1

Each number is the sum of the two immediately above it, taking zeros above when no number exist. The numbers are the detached coefficients in the expansion of $(1+x)^n$.

Early in the present century John Stone in the United States pointed out that by applying the binomial coefficients to the amplitudes of an antenna array, the minor lobes could be eliminated. The associated beam broadening is, however, unacceptable. In the 1940's analytical work in the United States and England established that beams sharper than binomial beams could be obtained with small minor lobes if Trinomial, Quadrinomial, Quinquenomial, etc., that is, "Multinomial" coefficients described the amplitude excitations. Selection of an appropriate multinomial set would provide a relation between beamwidth and minor lobe level. This work was superseded when, shortly afterward, Lieutenant C. L. Dolph of the U.S. Navy announced his remarkable theorem on the existence of optimum array coefficients. The intrinsic interest of the multinomial coefficients, however, and the relatively unfamiliar arithmetical triangles they led to, suggested the present note.

The Trinomial Triangle portrays the detached coefficients of the expansion $(1+x+x^2)^n$ and has the form

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		1	1	1			
		1	2	3	2	1	
		1	3	6	7	6	3
		1	4	10	16	19	16

Each number is the sum of the three immediately above it.

The Quadrinomial Triangle applying to $(1+x+x^2+x^3)^n$ takes the initial form

			1							
		1	2	3	4	3	2	1		
	1	3	6	10	12	12	10	6	3	1

and so on, each number being the sum of the four above it.

The radiation patterns of arrays having multinomial coefficients are, with $u = \pi (d/\lambda) \sin \theta$

Binomial	$2^n \cos^n u$
Trinomial	$(1+2 \cos 2u)^n$
Quadrinomial	$2^n (\cos u + \cos 3u)^n$
Quinquenomial	$[1+2(\cos 2u + \cos 4u)]^n$

In antenna theory, the antiphase aperture plays as important a role as the equiphase aperture; the excitation coefficients are then antisymmetrical functions instead of symmetrical functions. If, in a multinomial triangle, the vertical axis of symmetry contains zeros, with the coefficients to the right being positive and those to the left being negative, antisymmetrical or "odd" patterns are generated by arrays given these reversed phase coefficients. Such patterns can be regarded as produced by triangles obtained by multiplying each row of a multinomial triangle by a step operator,

If a multinomial triangle is operated upon by a linear operator, Fourier transform theory declares that the resulting antenna pattern will be the derivative or slope of the original pattern. Consider the antiphase binomial triangle of the form

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		-1	1		
	-1	0	1		
	-1	-1	1	1	
	-1	-2	0	2	1
-1	-3	-2	2	3	1

where again each number is the sum of the two above it. The resulting radiation patterns can be compared with the corresponding binomial patterns.

SYMMETRICAL PATTERNS

$2 \cos u$
$2^2 \cos^2 u$
$2^3 \cos^3 u$
$2^4 \cos^4 u$

ANTISYMMETRICAL "SLOPE" PATTERNS

$-2 \sin u$
$-2^2 \cos u \sin u$
$-2^3 \cos^2 u \sin u$
$-2^4 \cos^3 u \sin u$

Thus the odd functions are the derivatives of the even functions divided by n ; this antiphase triangle may be called a "slope generating triangle."

The slope generating triangle for a trinomial array is:

initial array is:

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			1	0	1					
		1	1	0	1	1				
	-1	2	2	0	2	2	1			
	-1	3	5	4	0	4	5	3	1	
-1	4	9	12	9	0	9	12	9	4	1

In the second row $n=2$ and the even pattern of the equiphase trinomial triangle is

$$(1+2 \cos 2u)^2$$

having a derivative

$$2(1+2 \cos 2u) (-4 \sin 2u) = -8 \sin 2u -8 \sin 4u$$

But the pattern due to $-1-1$ 0 1 1 is $-2 \sin 2u -2 \sin 4u$

which is the slope pattern divided by 4.

We conclude with the start of the slope generating triangle for quadrinomial arrays

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				3	1	1	3			
		3	4	3	0	3	4	3		
	3	7	10	10	4	4	10	10	7	3

obtained by adding in fours for each coefficient as for the symmetrical quadrinomial triangle.

The examples of numerical triangles show both the simplicity and the utility of these unusual mathematical forms in the antenna array context. It would be interesting to know whether the triangle technique applies further to more modern antenna aperture synthesis. Doubtless the triangles, whether as numerical, algebraic or complex matrices, have significance in probability theory, spectral theory or in distribution theory generally.