

Enumerative combinatorial problems concerning structures

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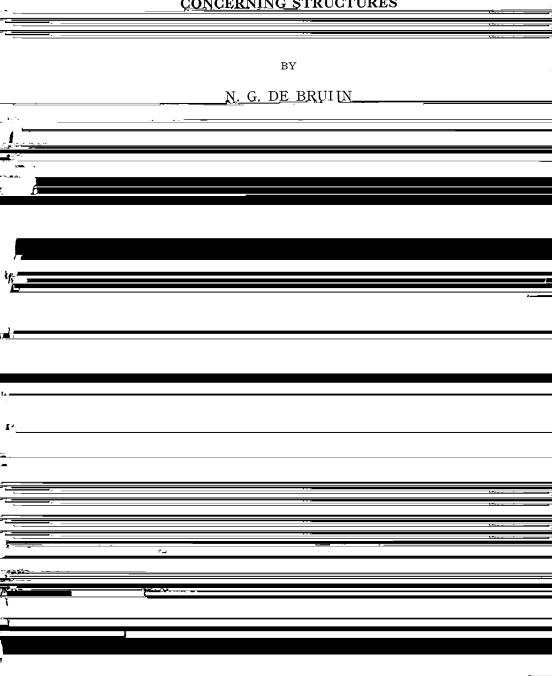
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Nieuw Archief voor Wiskunde (3), XI, 142-161 (1963)

ENUMERATIVE COMBINATORIAL PROBLEMS <u>CONCERNING STRUCTURES</u>



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D, R, S, T are sets, if $f \in R^D$, $g \in S^T$, and if $R \subset T$, then the compo-

sition gf is the mapping of D into S, defined as follows: (gf)(d) =

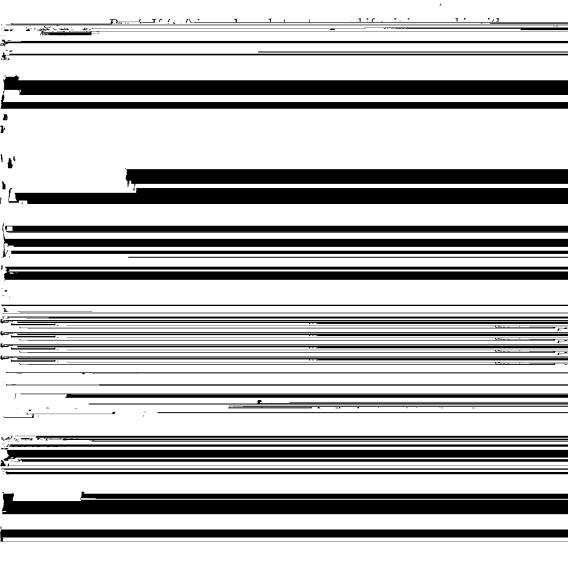
 $= g\{\underline{t(d)}\}$ for all $d \in D$. If A and B are sets, then $A \times B$ is the

is a finite set, then |D| represents the number of elements of D.

2. Structures on a tinite set. We shall not define here explicitly

what we mean by the word "structure". We shall only assume that

	ploured structures (although it is the base set D that is coloured	
	nd not the structure).	
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In other words, $Z_K = |G|^{-1} \sum_s \sum_g \psi(g)$, where the summation runs over all pairs is σ with $s \in K$, $\sigma \in G$, $\sigma(\sigma)_s = s$. Carrying out sum-

mation with respect to K, we obtain that

$$U = \sum_K Z_K = |G|^{-1} \sum_g \sum_s \psi(g);$$

where the summation is now restricted by $s \in S$, $g \in G$, $\sigma(g)s = s$.

If g is fixed, the number of possible s equals V(g), so our proof is complete.

We close this section by indicating an application of the polynomial U that is not a direct consequence of Pólya's theorem. We take a set of two colours. A structure s is said to be symmetrically

bicoloured if there is an automorphism of the structure that inter-

3. Examples with symmetric group G. In each example we take for S the set of all structures of a given type, and G is always the full symmetric group of D. The number of elements of D is called

n. In all examples of this section the U-polynomial will depend on n, and will be interpreted as the coefficient of w^n in a generating

function

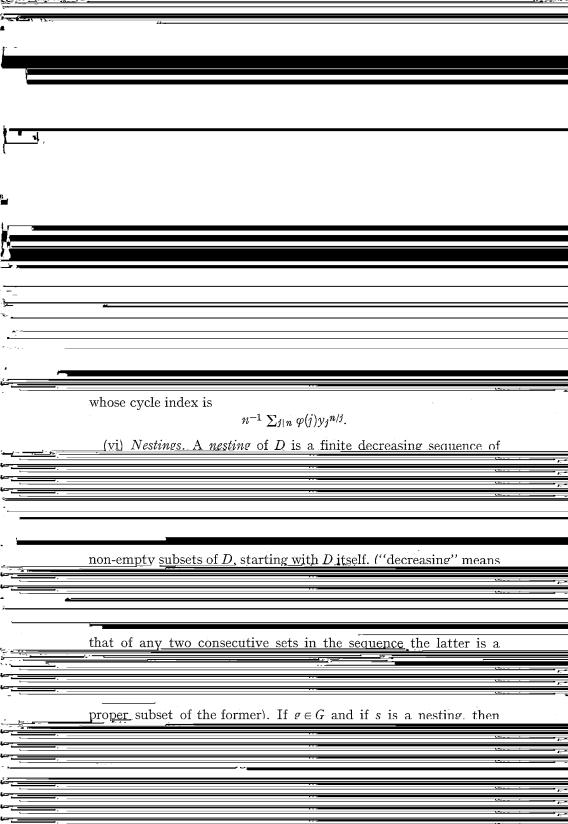
$$U(w; y_1, y_2, ...) = \sum_{0}^{\infty} w^n U_n(y_1, y_2, ...).$$
 (3.1)

(i) "Trivial" structures. Having the trivial structure on D means

that D is considered as just a set. Or, rather, the set S consists of

only one element, and the representation σ can be only the trivial one. So there is only one structure class, and the automorphism group of the one element in that class is G itself. So $U(y_1, y_2, ...) = P_G(y_1, y_2, ...)$. As G is the symmetric group of degree n, we know (see [7, 2]) that (3.1) equals

-	but we again suppress the proof. as (3.3) does not lead to new re-
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	sults. For colouring an ordered couple with colours taken from the
-	colour set R , can also be described as colouring the first component
	of the couple with a colour taken from the set $R \times R$.
	or the count with a colour taken from the set N ∧ N.
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	$s \in S$, $g \in G$, we define $\sigma(g)s = gsg^{-1}$ (so if s carries d into d', then





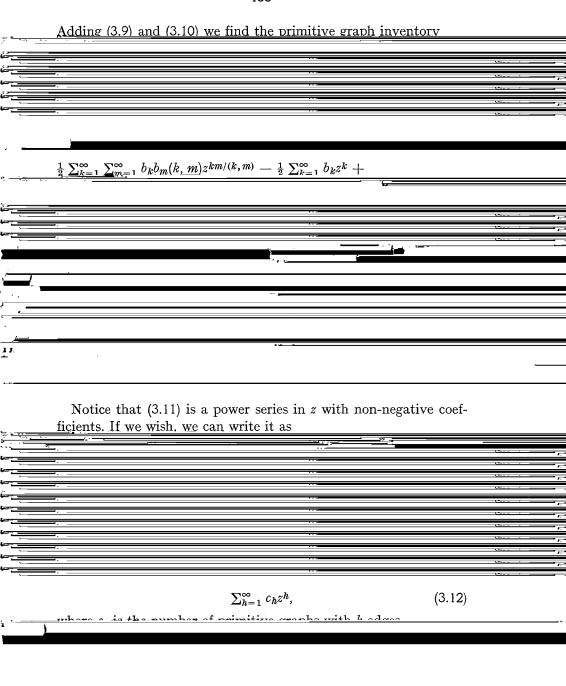
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$$\frac{1}{2}(n^2 - n)$$
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Using the notation (k, m) for the greatest common divisor of k and m, we shall show that

$$V(z; b_1, b_2, ...) = \prod_{k=1}^{\infty} \prod_{m=1}^{\infty} (1 + z^{km(k, m)})^{\frac{1}{2}(k, m)b_k b_m} \times \underbrace{\Pi^{\infty} \quad (1 + z^k)^{-\frac{1}{2}b_k} \Pi^{\infty}} \left\{ \underbrace{(1 + z^r)^2}_{2} \right\}^{\frac{1}{2}b_{2r}}$$

In order to prove this, we take a special permutation g of G, with $b_1(g) = b_1$, $b_2(g) = b_2$, etc. We want to count the number of graphs (with D as the set of nodes) which are invariant under $\sigma(g)$, or



4. Examples with a general group G. We again take a finite set D.

and a permutation group G of D, which is, in contrast to the previous section, not necessarily the symmetric group.

(viii) TH-structures. Let T be another finite set, and let H be a group of permutations of T. If both f_1 and f_2 are mappings of T into D, then they are called equivalent if and only if $f_1 = f_2h$ for some $h \in H$. The equivalence classes defined by this equivalence

will be called TH-structures. If $f \in D^T$, then the TH-structure to

f with gf = fh can be shown to be (see [2])

$$b_1^{c_1}(b_1 + 2b_2)^{c_2}(b_1 + 3b_3)^{c_3}(b_1 + 2b_2 + 4b_4)^{c_4} \dots =$$

$$= \left(\frac{\partial}{\partial z_1}\right)^{c_1} \left(\frac{\partial}{\partial z_2}\right)^{c_2} \dots \exp\left\{\sum_{j=1}^{\infty} j b_j (z_j + z_{2j} + z_{3j} + \dots)\right\}, \quad (4.3)$$

evaluated at $z_1 = z_2 = \dots = 0$. Taking the sum over all $h \in H$, and

dividing by
$$|H|$$
, we get $V(g)$. In order to get $U(v_1, v_2, ...)$ we apply

theorem 2, and that produces (4.1).

The special case $v_1 = v_2 = ... = 1$ in (4.2) reproduces a result of

[1, 2]. For, U(1, 1, 1, ...) is nothing but the number of structure

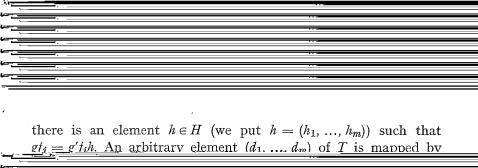
be interchanged without altering the colour scheme essentially. In required number is $P_H(1, 3, 1, 3,) = 9$. These nine solutions are easily obtained experimentally: one is all-white; two have one red		
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a class tH, then it contains the class gtH, for each $g \in G$. We shall

define the mapping ψ of S_1 onto S by

$$\psi(gf_jH) = \sigma_0(g)s_j, \tag{4.5}$$

but we have to show first that this definition is unambiguous. As-



(ix) Colourings. We take a set R of colours, and the structures

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	colours from R , i.e. $S = R^D$. And, for $g \in G$, we define $\sigma(g) = \sigma(g)t = tg^{-1}$ ($t \in R^D$). This is the same situation as in example	
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	(sec. 3), this time without restriction to the symmetric group as	nd
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Denoting $b_i(g)$ by b_i , $b_i(l)$ by c_i , the number of t with $tg^{-1} = lt$

$$\left(\frac{\partial}{\partial z_1}\right)^{b_1} \left(\frac{\partial}{\partial z_2}\right)^{b_2} \dots \exp\left\{\sum_{j=1}^{\infty} j c_j (z_j + z_{2j} + z_{3j} + \dots)\right\},$$

evaluated at $z_1 = z_2 = ... = 0$. In order to get $U(y_1, y_2, ...)$, we have to take the average over all $l \in L$, to multiply by $y_1^{b_1}y_2^{b_2}...$,

and to take the average over all
$$e \in G$$
. That leads to (4.6).

If the group L consists of the identity only,
$$P_L(x_1, x_2, ...)$$
 reduces

Similar simple results can always be obtained in situations where

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theorem (see [8, 9]). That theorem is obtained from the above result by taking each structure set to consist of a single structure

class only (whence the U's become cycle indexes), and putting

 $y_1 = y_2 = \dots = 1$ in the final result.

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