Enumerative combinatorial problems concerning structures

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ENUMERATIVE COMBINATORIAL PROBLEMS CONCERNING STRUCTURES

BY

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attempt was made to do something with a whole class of groups [redacted]

This is connected with the fact that there are not many classes of [redacted]

D, R, S, T are sets, if $f \in R^D$, $g \in S^T$, and if $R \subset T$, then the composition gf is the mapping of D into S, defined as follows: $(gf)(d) = g\{f(d)\}$ for all $d \in D$. If A and B are sets, then $A \times B$ is the

is a finite set, then $|D|$ represents the number of elements of D.

2. *Structures on a finite set*. We shall not define here explicitly

what we mean by the word "structure". We shall only assume that *coloured structures* (although it is the base set D that is coloured

and not the structure).

of structures, as we can introduce a representation σ^* of G as permutations of [illegible] S^*. If $g \in G$, we define $\sigma^*(g)$ by

In other words, $Z_K = |G|^{-1} \sum_s \sum_g \psi(g)$, where the summation runs over all pairs (s, g) with $s \in K$, $g \in G$, $\sigma(g)s = s$. Carrying out summation with respect to K, we obtain that

$$U = \textstyle\sum_K Z_K = |G|^{-1} \sum_g \sum_s \psi(g);$$

where the summation is now restricted by $s \in S$, $g \in G$, $\sigma(g)s = s$.

If g is fixed, the number of possible s equals $V(g)$, so our proof is complete.

We close this section by indicating an application of the polynomial U that is not a direct consequence of Pólya's theorem. We take a set of two colours. A structure s is said to be *symmetrically bicoloured* if there is an automorphism of the structure that inter-

3. *Examples with symmetric group G.* In each example we take for S the set of all structures of a given type, and G is always the full symmetric group of D. The number of elements of D is called n. In all examples of this section the U-polynomial will depend on n, and will be interpreted as the coefficient of w^n in a generating function

$$U(w; y_1, y_2, \ldots) = \sum_0^\infty w^n U_n(y_1, y_2, \ldots). \tag{3.1}$$

(i) *"Trivial" structures.* Having the trivial structure on D means that D is considered as just a set. Or, rather, the set S consists of only one element, and the representation σ can be only the trivial one. So there is only one structure class, and the automorphism group of the one element in that class is G itself. So $U(y_1, y_2, \ldots) = P_G(y_1, y_2, \ldots)$. As G is the symmetric group of degree n, we know (see [7, 2]) that (3.1) equals but we again suppress the proof, as (3.3) does not lead to new re-

sults. For colouring an ordered couple with colours taken from the colour set R, can also be described as colouring the first component

of the couple with a colour taken from the set $R \times R$.

$s \in S$, $g \in G$, we define $\sigma(g)s = gsg^{-1}$ (so if s carries d into d', then whose cycle index is

$$n^{-1} \sum_{j \mid n} \varphi(j) y_j^{n/j}.$$

(vi) *Nestings.* A *nesting* of D is a finite decreasing sequence of non-empty subsets of D, starting with D itself. ("decreasing" means that of any two consecutive sets in the sequence the latter is a proper subset of the former). If $g \in G$ and if s is a nesting, then leads to the final result

[redacted]

edges cannot exceed $\frac{1}{2}(n^2 - n)$).

[redacted]

Using the notation (k, m) for the greatest common divisor of k and m, we shall show that

$$V(z; b_1, b_2, \ldots) = \prod_{k=1}^{\infty} \prod_{m=1}^{\infty} (1 + z^{km(k,m)})^{\frac{1}{2}(k,m)b_k b_m} \times$$

[redacted]

In order to prove this, we take a special permutation g of G, with $b_1(g) = b_1$, $b_2(g) = b_2$, etc. We want to count the number of graphs (with D as the set of nodes) which are invariant under $\sigma(g)$, or

[redacted]

Adding (3.9) and (3.10) we find the primitive graph inventory

$$\tfrac{1}{2} \textstyle\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} b_k b_m (k, m) z^{km/(k,m)} - \tfrac{1}{2} \sum_{k=1}^{\infty} b_k z^k +$$

Notice that (3.11) is a power series in z with non-negative coefficients. If we wish, we can write it as

$$\textstyle\sum_{h=1}^{\infty} c_h z^h, \qquad (3.12)$$

where c_h is the number of primitive graphs with h edges.

4. *Examples with a general group G.* We again take a finite set D.

and a permutation group G of D, which is, in contrast to the previous section, not necessarily the symmetric group.

(viii) *TH-structures.* Let T be another finite set, and let H be a group of permutations of T. If both f_1 and f_2 are mappings of T into D, then they are called equivalent if and only if $f_1 = f_2 h$ for some $h \in H$. The equivalence classes defined by this equivalence

will be called TH-structures. If $f \in D^T$, then the TH-structure to

f with $gf = fh$ can be shown to be (see [2])

[redacted]

$$b_1^{c_1}(b_1 + 2b_2)^{c_2}(b_1 + 3b_3)^{c_3}(b_1 + 2b_2 + 4b_4)^{c_4} \dots =$$
$$= \left(\frac{\partial}{\partial z_1}\right)^{c_1}\left(\frac{\partial}{\partial z_2}\right)^{c_2} \dots \exp\{\textstyle\sum_{j=1}^{\infty} jb_j(z_j + z_{2j} + z_{3j} + \dots)\}, \quad (4.3)$$

evaluated at $z_1 = z_2 = \dots = 0$. Taking the sum over all $h \in H$, and

[redacted]

dividing by $|H|$, we get $V(g)$. In order to get $U(\nu_1, \nu_2, \dots)$ we apply

[redacted]

theorem 2, and that produces (4.1).

The special case $\nu_1 = \nu_2 = \dots = 1$ in (4.2) reproduces a result of

[redacted]

[1, 2]. For, $U(1, 1, 1, \dots)$ is nothing but the number of structure

[redacted]

classes, and these structure classes can also be interpreted as

[redacted]

A further question is, in how many ways we can colour the six faces with red, white and blue in such a way that red and blue can

be interchanged without altering the colour scheme essentially. In

required number is $P_H(1, 3, 1, 3, \ldots) = 9$. These nine solutions are easily obtained experimentally: one is all-white; two have one red a class fH, then it contains the class gfH, for each $g \in G$. We shall

[illegible]

define the mapping ψ of S_1 onto S by

$$\psi(gf_jH) = \sigma_0(g)s_j, \tag{4.5}$$

but we have to show first that this definition is unambiguous. As-

[illegible]

there is an element $h \in H$ (we put $h = (h_1, \ldots, h_m)$) such that $gf_j = g'f_ih$. An arbitrary element $(d_1, \ldots, d_m)$ of T is mapped by

[illegible]

(ix) *Colourings*. We take a set R of colours, and the structures to be considered in this example are just the colourings of D with colours from R, i.e. $S = R^D$. And, for $g \in G$, we define $\sigma(g)$ by $\sigma(g)f = fg^{-1}$ $(f \in R^D)$. This is the same situation as in example ii (sec. 3), this time without restriction to the symmetric group and

Denoting $b_i(g)$ by b_i, $b_i(l)$ by c_i, the number of f with $fg^{-1} = lf$

[illegible]

becomes (cf. (4.3))

$$\left(\frac{\partial}{\partial z_1}\right)^{b_1}\left(\frac{\partial}{\partial z_2}\right)^{b_2} \dots \exp\{\textstyle\sum_{j=1}^{\infty} jc_j(z_j + z_{2j} + z_{3j} + \dots)\},$$

evaluated at $z_1 = z_2 = \dots = 0$. In order to get $U(y_1, y_2, \dots)$, we have to take the average over all $l \in L$, to multiply by $y_1^{b_1} y_2^{b_2} \dots$,

[illegible]

and to take the average over all $g \in G$. That leads to (4.6).

[illegible]

If the group L consists of the identity only, $P_L(x_1, x_2, \dots)$ reduces

[illegible]

Similar simple results can always be obtained in situations where

of nestings (see sec. 3, example vi), we obtain theorem (see [8, 9]). That theorem is obtained from the above result by taking each structure set to consist of a single structure

class only (whence the U's become cycle indexes), and putting

$y_1 = y_2 = \ldots = 1$ in the final result.

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