

Enumerative combinatorial problems concerning structures

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ENUMERATIVE COMBINATORIAL PROBLEMS
CONCERNING STRUCTURES

BY

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attempt was made to do something with a whole class of groups

This is connected with the fact that there are not many classes of

D, R, S, T are sets, if $t \in R^D$, $g \in S^T$, and if $R \subset T$, then the compo-

sition gf is the mapping of D into S , defined as follows: $(gf)(d) =$
 $= g\{f(d)\}$ for all $d \in D$. If A and B are sets, then $A \times B$ is the

is a finite set, then $|D|$ represents the number of elements of D .

2. *Structures on a finite set.* We shall not define here explicitly

what we mean by the word "structure". We shall only assume that

coloured structures (although it is the base set D that is coloured

and not the structure).

of structures, as we can introduce a representation σ^* of G as permutations on S^* . If $g \in G$, we define $\sigma^*(g)$ by

In other words, $Z_K = |G|^{-1} \sum_s \sum_g \psi(g)$, where the summation runs over all pairs (s, g) with $s \in K$, $g \in G$, $g(g)s = s$. Carrying out sum-

mation with respect to K , we obtain that

$$U = \sum_K Z_K = |G|^{-1} \sum_g \sum_s \psi(g);$$

where the summation is now restricted by $s \in S$, $g \in G$, $g(g)s = s$.

If g is fixed, the number of possible s equals $V(g)$, so our proof is complete.

We close this section by indicating an application of the polynomial U that is not a direct consequence of Pólya's theorem. We take a set of two colours. A structure s is said to be *symmetrically*

bicoloured if there is an automorphism of the structure that inter-

3. *Examples with symmetric group G .* In each example we take for S the set of all structures of a given type, and G is always the full symmetric group of D . The number of elements of D is called

n . In all examples of this section the U -polynomial will depend on n , and will be interpreted as the coefficient of w^n in a generating

function

$$U(w; y_1, y_2, \dots) = \sum_0^\infty w^n U_n(y_1, y_2, \dots). \quad (3.1)$$

(i) "*Trivial*" structures. Having the trivial structure on D means

that D is considered as just a set. Or, rather, the set S consists of

only one element, and the representation σ can be only the trivial one. So there is only one structure class, and the automorphism group of the one element in that class is G itself. So $U(y_1, y_2, \dots) = P_G(y_1, y_2, \dots)$. As G is the symmetric group of degree n , we know (see [7, 2]) that (3.1) equals

but we again suppress the proof, as (3.3) does not lead to new re-

sults. For colouring an ordered couple with colours taken from the colour set R , can also be described as colouring the first component

of the couple with a colour taken from the set $R \times R$.

$s \in S$, $g \in G$, we define $\sigma(g)s = gsg^{-1}$ (so if s carries d into d' , then

whose cycle index is

$$n^{-1} \sum_{j|n} \varphi(j) y_j^{n/j}.$$

(vi) *Nestings*. A *nesting* of D is a finite decreasing sequence of

non-empty subsets of D , starting with D itself. ("decreasing" means

that of any two consecutive sets in the sequence, the latter is a

proper subset of the former). If $g \in G$ and if s is a nesting, then

leads to the final result

(needless to say $|J_n| = 0$ from a certain i onward, as the number of

edges cannot exceed $\frac{1}{2}(n^2 - n)$.

Using the notation (k, m) for the greatest common divisor of k and m , we shall show that

$$V(z; b_1, b_2, \dots) = \prod_{k=1}^{\infty} \prod_{m=1}^{\infty} (1 + z^{km(k, m)})^{\frac{1}{2}(k, m)b_k b_m} \times \prod_{k=1}^{\infty} (1 + z^{k^2})^{-\frac{1}{2}b_k} \prod_{r=1}^{\infty} \left\{ \frac{(1 + z^r)^2}{1 + z^{2r}} \right\}^{\frac{1}{2}b_{2r}} \quad (2.2)$$

In order to prove this, we take a special permutation g of G , with $b_1(g) = b_1$, $b_2(g) = b_2$, etc. We want to count the number of graphs (with D as the set of nodes) which are invariant under $\sigma(g)$, or

Adding (3.9) and (3.10) we find the primitive graph inventory

$$\frac{1}{2} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} b_k b_m (k, m) z^{km/(k, m)} - \frac{1}{2} \sum_{k=1}^{\infty} b_k z^k +$$

Notice that (3.11) is a power series in z with non-negative coefficients. If we wish, we can write it as

$$\sum_{h=1}^{\infty} c_h z^h, \quad (3.12)$$

where c_h is the number of primitive graphs with h edges

4. *Examples with a general group G .* We again take a finite set D .

and a permutation group G of D , which is, in contrast to the previous section, not necessarily the symmetric group.

(viii) *TH-structures.* Let T be another finite set, and let H be a group of permutations of T . If both f_1 and f_2 are mappings of T into D , then they are called equivalent if and only if $f_1 = f_2 h$ for some $h \in H$. The equivalence classes defined by this equivalence

will be called *TH-structures*. If $t \in D^T$, then the *TH-structure* to

t with $gt = fh$ can be shown to be (see [2])

$$b_1^{c_1}(b_1 + 2b_2)^{c_2}(b_1 + 3b_3)^{c_3}(b_1 + 2b_2 + 4b_4)^{c_4} \dots = \\ = \left(\frac{\partial}{\partial z_1} \right)^{c_1} \left(\frac{\partial}{\partial z_2} \right)^{c_2} \dots \exp \{ \sum_{j=1}^{\infty} j b_j (z_j + z_{2j} + z_{3j} + \dots) \}, \quad (4.3)$$

evaluated at $z_1 = z_2 = \dots = 0$. Taking the sum over all $h \in H$ and

dividing by $|H|$, we get $V(g)$. In order to get $U(v_1, v_2, \dots)$ we apply

theorem 2, and that produces (4.1).

The special case $v_1 = v_2 = \dots = 1$ in (4.2) reproduces a result of

[1, 2]. For, $U(1, 1, 1, \dots)$ is nothing but the number of structure

classes, and these structure classes can also be interpreted as

A further question is, in how many ways we can colour the six faces with red, white and blue in such a way that red and blue can

be interchanged without altering the colour scheme essentially. In

required number is $P_H(1, 3, 1, 3, \dots) = 9$. These nine solutions are easily obtained experimentally: one is all-white; two have one red

a class tH , then it contains the class gtH , for each $g \in G$. We shall

define the mapping ψ of S_1 onto S by

$$\psi(gt_jH) = \sigma_0(g)s_j, \quad (4.5)$$

but we have to show first that this definition is unambiguous. As-

there is an element $h \in H$ (we put $h = (h_1, \dots, h_m)$) such that $gt_j = g't_ih$. An arbitrary element (d_1, \dots, d_m) of T is mapped by

(ix) *Colourings*. We take a set R of colours, and the structures

to be considered in this example are just the colourings of D with

colours from R , i.e. $S = R^D$. And, for $g \in G$, we define $\sigma(g)$ by $\sigma(g)t = tg^{-1}$ ($t \in R^D$). This is the same situation as in example ii

(sec. 3), this time without restriction to the symmetric group and

Denoting $h_i(g)$ by h_i , $h_i(l)$ by c_i , the number of f with $fg^{-1} = lf$

becomes (cf. (4.3))

$$\left(\frac{\partial}{\partial z_1}\right)^{b_1} \left(\frac{\partial}{\partial z_2}\right)^{b_2} \dots \exp \left\{ \sum_{j=1}^{\infty} j c_j (z_j + z_{2j} + z_{3j} + \dots) \right\},$$

evaluated at $z_1 = z_2 = \dots = 0$. In order to get $U(y_1, y_2, \dots)$, we have to take the average over all $l \in L$, to multiply by $\gamma_1^{b_1} \gamma_2^{b_2} \dots$,

and to take the average over all $g \in G$. That leads to (4.6).

If the group L consists of the identity only, $P_l(x_1, x_2, \dots)$ reduces

Similar simple results can always be obtained in situations where

of nestings (see sec. 3 example vi), we obtain

theorem (see [8, 9]). That theorem is obtained from the above result by taking each structure set to consist of a single structure

class only (whence the U 's become cycle indexes), and putting

$y_1 = y_2 = \dots = 1$ in the final result.

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