THE

LONDON, EDINBURGH AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[FOURTH SERIES.]

MARCH 1861.

XXV. *On Physical Lines of Force.* By J. C. Maxwell, *Professor of Natural Philosophy in King's College, London**.

Part I.—*The Theory of Molecular Vortices applied to Magnetic*

[redacted]

Phenomena.

In all phenomena involving attractions or repulsions, or any forces depending on the relative position of bodies, we have to determine the *magnitude* and *direction* of the force which would act on a given body, if placed in a given position.

[redacted]

the lines of force as something real, and as indicating something more than the mere resultant of two forces, whose seat of action is at a distance, and which do not exist there at all until a magnet is placed in that part of the field. We are dissatisfied with the explanation founded on the hypothesis of attractive and repellent forces directed towards the magnetic poles, even though we may have satisfied ourselves that the phenomenon is in strict accordance with that hypothesis, and we cannot help thinking that in every place where we find these lines of force, some physical state or action must exist in sufficient energy to produce the actual phenomena.

My object in this paper is to clear the way for speculation in this direction, by investigating the mechanical results of certain states of tension and motion in a medium, and comparing these with the observed phenomena of magnetism and electricity. By pointing out the mechanical consequences of such hypotheses, I hope to be of some use to those who consider the phenomena as due to the action of a medium, but are in doubt as to the relation of this hypothesis to the experimental laws already established, which have generally been expressed in the language of other

The mechanical conditions of a medium under magnetic in-

The necessary relations among these forces have been investigated by mathematicians; and it has been shown that the most general type of a stress consists of a combination of three principal pressures or tensions, in directions at right angles to each other.

When two of the principal pressures are equal, the third becomes an axis of symmetry, either of greatest or least pressure, [redacted]

When the three principal pressures are equal, the pressure is [redacted] equal in every direction, and there results a stress having no determinate axis of direction, of which we have an example in simple hydrostatic pressure. [redacted] those near two magnetic poles of the same name; but we know that the mechanical effect is that of attraction instead of repulsion. The lines of force in this case do not run between the bodies, but avoid each other, and are dispersed over space. In traversing similar parts of the systems will be m; so that $l^3 m n$ is the ratio of the momenta acquired by similar portions in traversing similar parts of their paths.

The ratio of the surfaces is l^2, that of the forces acting on them is $l^2 p$, and that of the times during which they act is $\frac{l}{m}$; so that the ratio of the impulse of the forces is $\frac{l^3 p}{m}$, and we have now

$$l^3 m n = \frac{l^3 p}{m},$$

or

$$m^2 n = p;$$

that is, the ratio of the pressures due to the motion (p) is compounded of the ratio of the densities (n) and the duplicate ratio of the velocities (m^2), and does not depend on the linear dimensions of the moving systems.

In a circular vortex, revolving with uniform angular velocity, if the pressure at the axis is p_0, that at the circumference will be and its velocity to diminish in the same proportion. In order that a medium having these inequalities of pressure in different directions should be in equilibrium, certain conditions must be fulfilled, which we must investigate.

Prop. II.—If the direction-cosines of the axes of the vortices with respect to the axes of x, y, and z be l, m, and n, to find the normal and tangential stresses on the coordinate planes.

The actual stress may be resolved into a simple hydrostatic pressure p_1 acting in all directions, and a simple tension $p_1 - p_2$, or $\frac{1}{4\pi}\mu v^2$, acting along the axis of stress.

Hence if p_{xx}, p_{yy}, and p_{zz} be the normal stresses parallel to the three axes, considered positive when they tend to increase those axes; and if p_{yz}, p_{zx}, and p_{xy} be the tangential stresses in the three coordinate planes, considered positive when they tend

We have in general, for the force in the direction of x per unit of volume by the law of equilibrium of stresses*,

$$X=\frac{d}{dx}p_{xx}+\frac{d}{dy}p_{xy}+\frac{d}{dz}p_{xz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

[redacted]

$$X=\frac{1}{4\pi}\left\{\frac{d(\mu\alpha)}{dx}\alpha+\mu\alpha\frac{d\alpha}{dx}-4\pi\frac{dp_1}{dx}+\frac{d(\mu\beta)}{dy}\alpha+\mu\beta\frac{d\alpha}{dy}\right.$$
$$\left.+\frac{d(\mu\gamma)}{dz}\alpha+\mu\gamma\frac{d\alpha}{dz}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Remembering that $\alpha\frac{d\alpha}{dx}+\beta\frac{d\beta}{dx}+\gamma\frac{d\gamma}{dx}=\frac{1}{2}\frac{d}{dx}(\alpha^2+\beta^2+\gamma^2)$, this becomes

$$X=\alpha\frac{1}{4\pi}\left(\frac{d}{dx}(\mu\alpha)+\frac{d}{dy}(\mu\beta)+\frac{d}{dz}(\mu\gamma)\right)+\frac{1}{8\pi}\mu\frac{d}{dx}(\alpha^2+\beta^2+\gamma^2)$$

[redacted]

where α is the intensity of the magnetic force, and m is the amount of magnetic matter pointing north in unit of volume.

The physical interpretation of this term is, that the force on A will be to pull it more powerfully towards D than towards C; that is, A will tend to move to the north.

Let B in fig. 2 represent a south pole. The lines of force coming more numerous towards the right. It may be shown element will be urged in the direction of $-x$, transversely to the

The second term to the action on bodies capable of magnetism by induction.

The third and fourth terms to the force acting on electric currents.

And the fifth to the effect of simple pressure.

Before going further in the general investigation, we shall consider equations (12, 13, 14,) in particular cases, corresponding to those simplified cases of the actual phenomena which we seek

Hence the lines of force in a part of space where μ is uniform, and where there are no electric currents, must be such as would result from the theory of "imaginary matter" acting at a distance. The assumptions of that theory are unlike those of ours, but the results are identical.

Let us first take the case of a single magnetic pole, that is, one end of a long magnet, so long that its other end is too far off to have a perceptible influence on the part of the field we are considering. The conditions then are, that equation (18) must be fulfilled at the magnetic pole, and (19) everywhere else. The only solution under these conditions is

$$\phi = -\frac{m}{\mu}\frac{1}{r}, \quad \ldots \ldots \ldots \quad (20)$$

where r is the distance from the pole, and m the strength of the pole.

The repulsion at any point on a unit pole of the same kind is

$$\frac{d\phi}{dr} = \frac{m}{\mu}\frac{1}{r^2}. \quad \ldots \ldots \ldots \quad (21)$$

In the standard medium $\mu = 1$; so that the repulsion is simply $\frac{m}{r^2}$ in that medium, as has been shown by Coulomb.

In a medium having a greater value of μ (such as oxygen, solutions of salts of iron, &c.) the attraction, on our theory, ought to be *less* than in air, and in diamagnetic media (such as water, melted bismuth, &c.) the attraction between the same magnetic poles ought to be *greater* than in air.

The experiments necessary to demonstrate the difference of attraction of two magnets according to the magnetic or diamagnetic character of the medium in which they are placed, would require great precision, on account of the limited range of magnetic capacity in the fluid media known to us, and the small amount of the difference sought for as compared with the whole attraction.

Let us next take the case of an electric current whose quantity is C, flowing through a cylindrical conductor whose radius is R, and whose length is infinite as compared with the size of the field of force considered.

Let the axis of the cylinder be that of z, and the direction of so that within the conductor

$$\alpha=-2\frac{C}{R^2}y, \quad \beta=2\frac{C}{R^2}x, \quad \gamma=0. \quad . \quad . \quad . \quad . \quad (23)$$

Beyond the conductor, in the space round it,

$$\phi=2C\tan^{-1}\frac{y}{x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$\alpha=\frac{d\phi}{dx}=-2C\frac{y}{x^2+y^2}, \quad \beta=\frac{d\phi}{dy}=2C\frac{x}{x^2+y^2}, \quad \gamma=\frac{d\phi}{dz}=0. \ (25)$$

If $\rho=\sqrt{x^2+y^2}$ is the perpendicular distance of any point from the axis of the conductor, a unit north pole will experience a force $=\frac{2C}{\rho}$, tending to move it round the conductor in the direction of the hands of a watch, if the observer view it in the direction of the current.

Let us now consider a current running parallel to the axis of z in the plane of xz at a distance ρ. Let the quantity of the current be c', and let the length of the part considered be l, and its section s, so that $\frac{c'}{s}$ is its strength per unit of section. Putting this quantity for p in equations (12, 13, 14), we find

$$X=-\mu\beta\frac{c'}{s}$$

per unit of volume; and multiplying by ls, the volume of the conductor considered, we find

$$X=-\mu\beta c'l$$

$$=-2\mu\frac{Cc'l}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

showing that the second conductor will be attracted towards the first with a force inversely as the distance.

We find in this case also that the amount of attraction depends on the value of μ, but that it varies directly instead of inversely as μ; so that the attraction between two conducting wires will be greater in oxygen than in air, and greater in air than in water.

We shall next consider the nature of electric currents and electromotive forces in connexion with the theory of molecular vortices.

XLIV. *On Physical Lines of Force.* By J. C. MAXWELL, *Professor of Natural Philosophy in King's College, London*†.

[With a Plate.]

PART II.—*The Theory of Molecular Vortices applied to Electric Currents.*

WE have already shown that all the forces acting between magnets, substances capable of magnetic induction, and position that the surrounding medium is put into such a state that at every point the pressures are different in different directions, the direction of least pressure being that of the observed lines of force, and the difference of greatest and least pressures being proportional to the square of the intensity of the force at that point.

Such a state of stress, if assumed to exist in the medium, and to be arranged according to the known laws regulating lines of

We know that the lines of force are affected by electric cur-

rent; so that from the force we can determine the amount of the current. Assuming that our explanation of the lines of force by molecular vortices is correct, why does a particular distribution of vortices indicate an electric current? A satisfactory answer to this question would lead us a long way towards that of a very important one, "What is an electric current?"

I have found great difficulty in conceiving of the existence of

$$n\beta - m\gamma \quad \text{parallel to } x,$$
$$l\gamma - n\alpha \quad \text{parallel to } y,$$
$$m\alpha - l\beta \quad \text{parallel to } z.$$

If this portion of the surface be in contact with another vortex

$$u=\frac{1}{2}\frac{d\gamma}{dx}\Big(m_1(x-x_1)+m_2(x-x_2)\Big)$$
$$+\frac{1}{2}\frac{d\gamma}{dy}\Big(m_1(y-y_1)+m_2(y-y_2)\Big)+\frac{1}{2}\frac{d\gamma}{dz}\Big(m_1(z-z_1)+m_2(z-z_2)\Big)$$
$$-\frac{1}{2}\frac{d\beta}{dx}\Big(n_1(x-x_1)+n_2(x-x_2)\Big)-\frac{1}{2}\frac{d\beta}{dy}\Big(n_1(y-y_1)+n_2(y-y_2)\Big)$$
$$-\frac{1}{2}\frac{d\beta}{dz}\Big(n_1(z-z_1)+n_2(z-z_2)\Big) \quad . \quad . \quad (31)$$

In effecting the summation of $\Sigma up dS$, we must remember that round any closed surface $\Sigma l dS$ and all similar terms vanish; also that terms of the form $\Sigma l y dS$, where l and y are measured in different directions, also vanish; but that terms of the form $\Sigma l x dS$, where l and x refer to the same axis of coordinates, do not vanish, but are equal to the volume enclosed by the surface. The result is

$$\overline{V}p=\frac{1}{2}\rho\left(\frac{d\gamma}{dy}-\frac{d\beta}{dz}\right)(V_1+V_2+\&c.); \quad . \quad . \quad (32)$$

or dividing by $\overline{V}=V_1+V_2+\&c.$,

$$p=\frac{1}{2}\rho\left(\frac{d\gamma}{dy}-\frac{d\beta}{dz}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

If we make

$$\rho=\frac{1}{2\pi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

then equation (33) will be identical with the first of equations (9), which give the relation between the quantity of an electric current and the intensity of the lines of force surrounding it.

It appears therefore that, according to our hypothesis, an electric current is represented by the transference of the moveable particles interposed between the neighbouring vortices. We may conceive that these particles are very small compared with the size of a vortex, and that the mass of all the particles together is inappreciable compared with that of the vortices, and perience resistance, so as to waste electrical energy and generate heat.

Now let us suppose the vortices arranged in a medium in any arbitrary manner. The quantities $\frac{d\gamma}{dy}-\frac{d\beta}{dz}$, &c. will then in general have values, so that there will at first be electrical currents in the medium. These will be opposed by the electrical resistance of the medium; so that, unless they are kept up by a continuous supply of force, they will quickly disappear, and we shall then have $\frac{d\gamma}{dy}-\frac{d\beta}{dz}=0$, &c.; that is, $\alpha dx+\beta dy+\gamma dz$ will be a complete differential (see equations (15) and (16)); so that our hypothesis accounts for the distribution of the lines of force.

In Plate V. fig. 1, let the vertical circle E E represent an electric current flowing from copper C to zinc Z through the conductor E E′, as shown by the arrows.

Let the horizontal circle M M′ represent a line of magnetic force embracing the electric circuit, the north and south directions being indicated by the lines S N and N S.

Let the vertical circles V and V′ represent the molecular vortices of which the line of magnetic force is the axis. V revolves in unit of volume is

$$E = C\mu(\alpha^2+\beta^2+\gamma^2),$$

where C is a constant to be determined.

Let us take the case in which

$$\alpha=\frac{d\phi}{dx}, \quad \beta=\frac{d\phi}{dy}, \quad \gamma=\frac{d\phi}{dz}. \quad . \quad . \quad . \quad (35)$$

Let

$$\phi=\phi_1+\phi_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

have by the conservation of force,

$$\delta \mathrm{E} + \delta \mathrm{W} = 0 \; ; \quad \ldots \ldots \ldots \quad (43)$$

that is, the loss of energy of the vortices must be made up by work done in moving magnets, so that

$$-4\pi \mathrm{C}\Sigma\left(2\frac{d\phi_1}{dx} m_2 d\mathrm{V}\right)\delta x + \Sigma\left(\frac{d\phi_1}{dx} m_2 d\mathrm{V}\right)\delta x = 0,$$

or

$$\mathrm{C} = \frac{1}{8\pi} \; ; \quad \ldots \ldots \ldots \quad (44)$$

so that the energy of the vortices in unit of volume is

$$\frac{1}{8\pi}\mu(\alpha^2+\beta^2+\gamma^2) \; ; \quad \ldots \ldots \ldots \quad (45)$$

and that of a vortex whose volume is V is

$$\frac{1}{8\pi}\mu(\alpha^2+\beta^2+\gamma^2)\mathrm{V}. \quad \ldots \ldots \ldots \quad (46)$$

In order to produce or destroy this energy, work must be expended on, or received from, the vortex, either by the tangential action of the layer of particles in contact with it, or by change of form in the vortex. We shall first investigate the tangential action between the vortices and the layer of particles in contact with them.

Prop. VII.—To find the energy spent upon a vortex in unit of time by the layer of particles which surrounds it.

Let P, Q, R be the forces acting on unity of the particles in the three coordinate directions, these quantities being functions of x, y, and z. Since each particle touches two vortices at the extremities of a diameter, the reaction of the particle on the vortices will be equally divided, and will be

$$-\frac{1}{2}\mathrm{P}, \quad -\frac{1}{2}\mathrm{Q}, \quad -\frac{1}{2}\mathrm{R}$$

on each vortex for unity of the particles; but since the superficial density of the particles is $\frac{1}{2\pi}$ (see equation (34)), the forces on unit of surface of a vortex will be

$$-\frac{1}{4\pi}\mathrm{P}, \quad -\frac{1}{4\pi}\mathrm{Q}, \quad -\frac{1}{4\pi}\mathrm{R}.$$

Now let $d\mathrm{S}$ be an element of the surface of a vortex. Let the direction-cosines of the normal be l, m, n. Let the coordinates of the element be x, y, z. Let the component velocities of the

$$\frac{dE}{dt} = -\frac{1}{4\pi}(Pu + Qv + Rw)dS. \quad . \quad . \quad . \quad . \quad (47)$$

Let us begin with the first term, $Pu\,dS$. P may be written

$$P_0 + \frac{dP}{dx}x + \frac{dP}{dy}y + \frac{dP}{dz}z, \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

and

$$u = n\beta - m\gamma.$$

Remembering that the surface of the vortex is a closed one, so

$$\left.\begin{aligned} &\frac{dQ}{dz}-\frac{dR}{dy}=\mu\frac{d\alpha}{dt}. \\ \text{Similarly,}\quad &\frac{dR}{dx}-\frac{dP}{dz}=\mu\frac{d\beta}{dt}, \\ \text{and}\quad &\frac{dP}{dy}-\frac{dQ}{dx}=\mu\frac{d\gamma}{dt}. \end{aligned}\right\} \quad \ldots \ldots \quad (54)$$

From these equations we may determine the relation between the alterations of motion $\frac{d\alpha}{dt}$, &c. and the forces exerted on the layers of particles between the vortices, or, in the language of our hypothesis, the relation between changes in the state of the magnetic field and the electromotive forces thereby brought into play.

In a memoir "On the Dynamical Theory of Diffraction" (Cambridge Philosophical Transactions, vol. ix. part 1, section 6), Professor Stokes has given a method by which we may solve equations (54), and find P, Q, and R in terms of the quantities on the right-hand of those equations. I have pointed out* the application of this method to questions in electricity and magnetism.

Let us then find three quantities F, G, H from the equations

$$\left.\begin{aligned} \frac{dG}{dz}-\frac{dH}{dy}&=\mu\alpha, \\ \frac{dH}{dx}-\frac{dF}{dz}&=\mu\beta, \\ \frac{dF}{dy}-\frac{dG}{dx}&=\mu\gamma, \end{aligned}\right\} \quad \ldots \ldots \quad (55)$$

with the conditions

$$\frac{1}{4\pi}\left(\frac{d}{dx}\mu\alpha+\frac{d}{dy}\mu\beta+\frac{d}{dz}\mu\gamma\right)=m=0, \quad . \quad (56)$$

We have thus determined three quantities, F, G, H, from which we can find P, Q, and R by considering these latter quantities as the rates at which the former ones vary. In the paper already referred to, I have given reasons for considering the quantities F, G, H as the resolved parts of that which Faraday has conjectured to exist, and has called the *electrotonic state*. In that paper I have stated the mathematical relations between this electrotonic state and the lines of magnetic force as expressed in

Here the near coincidence of the results in the first and third columns shows that the relation between k and T may be approximately expressed by the formula

$$k = 14{\cdot}15\,\mathrm{T}^{\frac{1}{3}}, \text{ or } \mathrm{T} = \left(\frac{k}{14{\cdot}15}\right)^{3}. \quad . \quad . \quad . \quad (7)$$

Hastings, April 1, 1861.

LI. *On Physical Lines of Force.* By J. C. Maxwell, *Professor of Natural Philosophy in King's College, London.*

[With a Plate.]

Part II.—*The Theory of Molecular Vertices applied to Electric Currents.*

[Concluded from p. 291.]

As an example of the action of the vortices in producing induced currents, let us take the following case:—Let B, it is broken, there will be a current through C in the same direction as the primary current.

We may now perceive that induced currents are produced when the electricity yields to the electromotive force,—this force, however, still existing when the formation of a sensible current is prevented by the resistance of the circuit.

The electromotive force, of which the components are P, Q, R, arises from the action between the vortices and the interposed particles, when the velocity of rotation is altered in any part of the field. It corresponds to the pressure on the axle of a wheel in a machine when the velocity of the driving wheel is increased or diminished.

The electrotonic state, whose components are F, G, H, is what the electromotive force would be if the currents, &c. to which the lines of force are due, instead of arriving at their actual state by degrees, had started instantaneously from rest with their actual values. It corresponds to the *impulse* which would act on the axle of a wheel in a machine if the actual velocity were suddenly given to the driving wheel, the machine being previously at rest.

If the machine were suddenly stopped by stopping the driving wheel, each wheel would receive an impulse equal and opposite to that which it received when the machine was set in motion.

This impulse may be calculated for any part of a system of mechanism, and may be called the *reduced momentum* of the machine for that point. In the varied motion of the machine, the actual force on any part arising from the variation of motion may be found by differentiating the reduced momentum with any set of three axes. We shall first consider the effect of three simple extensions or compressions.

Prop. IX.—To find the variations of α, β, γ in the parallelopiped x, y, z when x becomes $x+\delta x$; y, $y+\delta y$; and z, $z+\delta z$;

along three axes properly chosen, x', y', z', the nine direction-cosines of these axes with their six connecting equations, which are equivalent to three independent quantities, and the three rotations θ_1, θ_2, θ_3 about the axes of x, y, z.

Let the direction-cosines of x' with respect to x, y, z be l_1, m_1, n_1, those of y', l_2, m_2, n_2, and those of z', l_3, m_3, n_3; then we find

$$\left.\begin{aligned}\frac{d}{dx}\delta x &= l_1^2\frac{\delta x'}{x'}+l_2^2\frac{\delta y'}{y'}+l_3^2\frac{\delta z'}{z'},\\ \frac{d}{dy}\delta x &= l_1m_1\frac{\delta x'}{x'}+l_2m_2\frac{\delta y'}{y'}+l_3m_3\frac{\delta z'}{z'}-\theta_3,\\ \frac{d}{dz}\delta x &= l_1n_1\frac{\delta x'}{x'}+l_2n_2\frac{\delta y'}{y'}+l_3n_3\frac{\delta z'}{z'}+\theta_2,\end{aligned}\right\} \quad . \quad . \quad (64)$$

with similar equations for quantities involving δy and δz.

Let α', β', γ' be the values of α, β, γ referred to the axes of x', y', z'; then

$$\left.\begin{aligned}\alpha' &= l_1\alpha+m_1\beta+n_1\gamma,\\ \beta' &= l_2\alpha+m_2\beta+n_2\gamma,\\ \gamma' &= l_3\alpha+m_3\beta+n_3\gamma.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

We shall then have

$$\delta\alpha = l_1\delta\alpha'+l_2\delta\beta'+l_3\delta\gamma'+\gamma\theta_2-\beta\theta_3 \quad . \quad . \quad . \quad (66)$$

[redacted]

Equating the two values of $\delta\alpha$ and dividing by δt, and remembering that in the motion of an incompressible medium

and that in the absence of free magnetism

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

we find

$$\frac{1}{\mu}\left(\frac{dQ}{dz}-\frac{dR}{dy}\right)+\gamma\frac{d}{dz}\frac{dx}{dt}-\alpha\frac{d}{dz}\frac{dz}{dt}-\alpha\frac{d}{dy}\frac{dy}{dt}+\beta\frac{d}{dy}\frac{dx}{dt}$$
$$+\frac{d\gamma}{dz}\frac{dx}{dt}-\frac{d\alpha}{dz}\frac{dz}{dt}-\frac{d\alpha}{dy}\frac{dy}{dt}+\frac{d\beta}{dy}\frac{dx}{dt}-\frac{d\alpha}{dt}=0. \quad . \quad (73)$$

Putting

$$\alpha=\frac{1}{\mu}\left(\frac{dG}{dz}-\frac{dH}{dy}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

and

$$\frac{d\alpha}{dt}=\frac{1}{\mu}\left(\frac{d^2G}{dz\,dt}-\frac{d^2H}{dy\,dt}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

where F, G, and H are the values of the electrotonic components for a fixed point of space, our equation becomes

$$\frac{d}{dz}\left(Q+\mu\gamma\frac{dx}{dt}-\mu\alpha\frac{dz}{dt}-\frac{dG}{dt}\right)-\frac{d}{dy}\left(R+\mu\alpha\frac{dy}{dt}-\mu\beta\frac{dx}{dt}-\frac{dH}{dt}\right)=0. \quad (76)$$

The expressions for the variations of β and γ give us two other

The physical meaning of the terms in the expression for the electromotive force depending on the motion of the body, may be made simpler by supposing the field of magnetic force uniformly magnetized with intensity α in the direction of the axis of x. Then if l, m, n be the direction-cosines of any portion of [redacted]

[redacted]

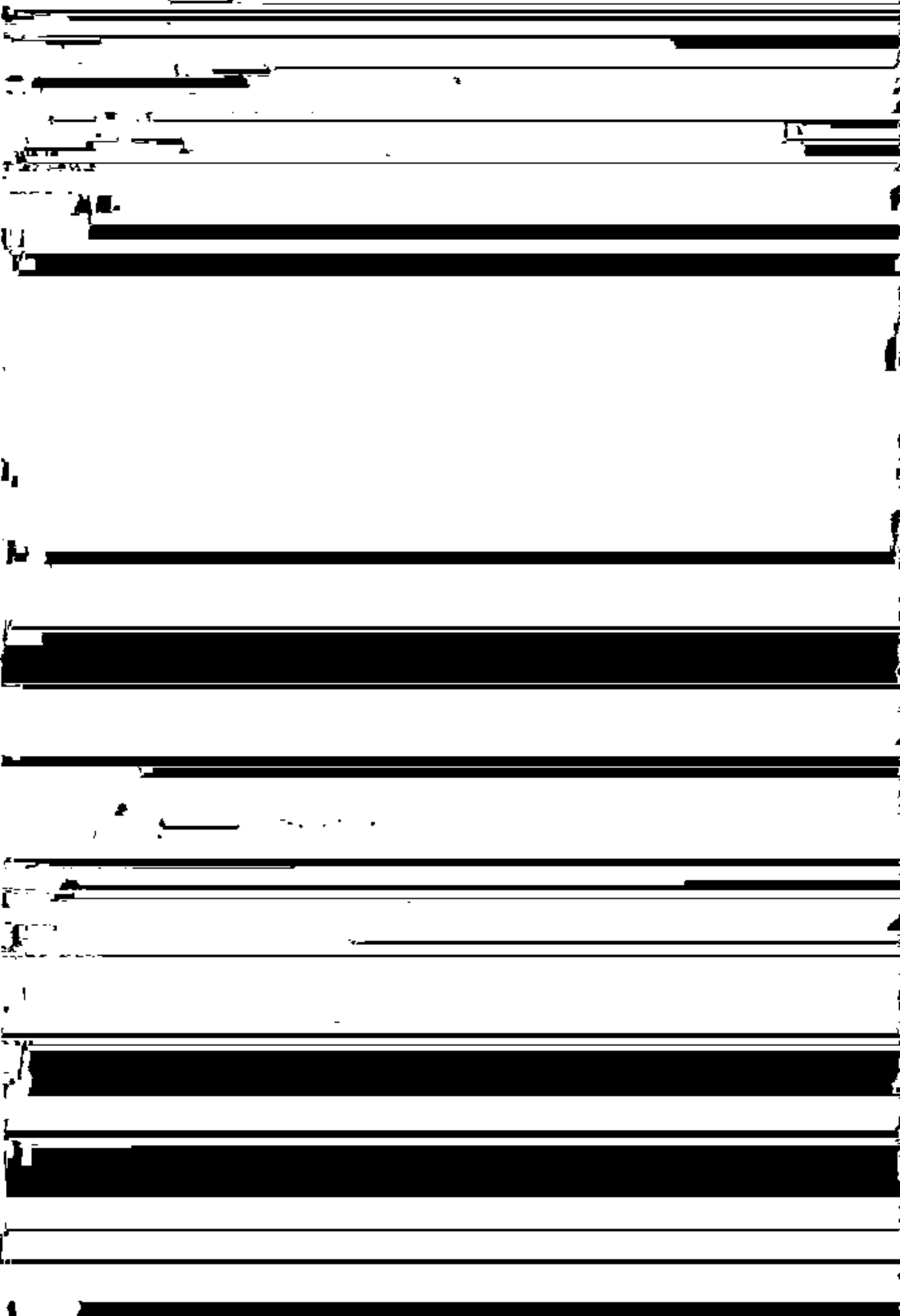
resolved in the direction of the conductor will be

$$e = \mathrm{S}(\mathrm{P}l + \mathrm{Q}m + \mathrm{R}n), \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

or

$$e = \mathrm{S}\mu\alpha\left(m\frac{dz}{dt} - n\frac{dy}{dt}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

that is, the product of $\mu\alpha$, the quantity of magnetic induction [redacted] motion relative to it. It is evident that, from this figure, we can trace the variations of form of an element of the fluid, as the form of the element depends, not on the absolute motion of the whole system, but on the relative motion of its parts.



ance to its motion. We may now recapitulate the assumptions we have made, and the results we have obtained.

(1) Magneto-electric phenomena are due to the existence of matter under certain conditions of motion or of pressure in every part of the magnetic field, and not to direct action at a distance between the magnets or currents. The substance producing these effects may be a certain part of ordinary matter, or it may

and so to change their place, provided they keep within one be moved in a direction contrary to that of the current, so that there will be an induced current in the opposite direction to the primary one.

If there were no resistance to the motion of the particles, the

The facts of electro-magnetism are so complicated and various, that the explanation of any number of them by several different hypotheses must be interesting, not only to physicists, but to all who desire to understand how much evidence the explanation of phenomena lends to the credibility of a theory, or how far we ought to regard a coincidence in the mathematical expression of two sets of phenomena as an indication that these phenomena are of the same kind. We know that partial coincidences of this kind have been discovered: and the fact that they are only

Fig. 1

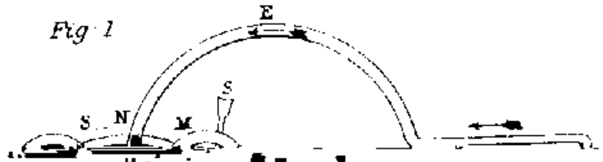
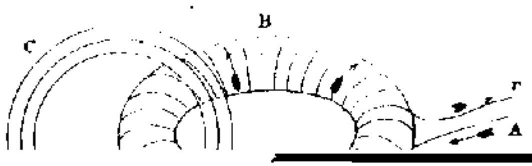


Fig. 2

Fig. 3



III. *On Physical Lines of Force.* By J. C. MAXWELL, *F.R.S., Professor of Natural Philosophy in King's College, London**.

PART III.—*The Theory of Molecular Vortices applied to Statical Electricity.*

IN the first part of this paper† I have shown how the forces acting between magnets, electric currents, and matter capable of magnetic induction may be accounted for on the hypothesis of the magnetic field being occupied with innumerable vortices

I conceived the rotating matter to be the substance of certain cells, divided from each other by cell-walls composed of particles not flow through them, electrical effects are propagated through them, and the amount of these effects differs according to the nature of the body; so that equally good insulators may act differently as dielectrics*.

Here then we have two independent qualities of bodies, one by which they allow of the passage of electricity through them, and the other by which they allow of electrical action being transmitted through them without any electricity being allowed to pass. A conducting body may be compared to a porous membrane which opposes more or less resistance to the passage

These relations are independent of any theory about the internal mechanism of dielectrics; but when we find electromotive force producing electric displacement in a dielectric, and when we find

Let μ be the coefficient of cubic elasticity, so that if

$$p_{xx}=p_{yy}=p_{zz}=p,$$

$$p=\mu\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right). \quad . \quad . \quad . \quad . \quad (80)$$

Let m be the coefficient of rigidity, so that

$$p_{xx}-p_{yy}=m\left(\frac{d\xi}{dx}-\frac{d\eta}{dy}\right), \text{ \&c.} \quad . \quad . \quad . \quad . \quad . \quad (81)$$

Then we have the following equations of elasticity in an isotropic medium,

$$p_{xx}=(\mu-\tfrac{1}{3}m)\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)+m\frac{d\xi}{dx}; \quad . \quad (82)$$

with similar equations in y and z, and also

$$p_{yz}=\frac{m}{2}\left(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\right), \text{ \&c.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

In the case of the sphere, let us assume the radius $=a$, and

$$\xi=exz, \quad \eta=exy, \quad \zeta=f(x^2+y^2)+gz^2+d. \quad . \quad (84)$$

Then

$$p_{xx}=2(\mu-\tfrac{1}{3}m)(e+g)z+mez=p_{yy},$$

The normal stress on the surface at any point is

$$N = p_{xx}\sin^2\theta + p_{yy}\cos^2\theta + 2p_{xz}\sin\theta\cos\theta$$

$$= 2(\mu - \tfrac{1}{3}m)(e+g)a\cos\theta + 2ma\cos\theta\big((e+f)\sin^2\theta + g\cos^2\theta\big); \quad (92)$$

or by (87) and (90),

$$N = -ma(e+2f)\cos\theta. \quad . \quad . \quad . \quad . \quad (93)$$

The tangential displacement of any point is

$$t = \xi\cos\theta - \zeta\sin\theta = -(a^2f+d)\sin\theta. \quad . \quad . \quad . \quad (94)$$

The normal displacement is

$$n = \xi\sin\theta + \zeta\cos\theta = (a^2(e+f)+d)\cos\theta. \quad . \quad . \quad (95)$$

[redacted]

$$a^2(e+f)+d=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

there will be no normal displacement, and the displacement will be entirely tangential, and we shall have

$$t = a^2e\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

The whole work done by the superficial forces is

$$U = \tfrac{1}{2}\Sigma(Tt)dS,$$

the summation being extended over the surface of the sphere.

The energy of elasticity in the substance of the sphere is

$$U = \tfrac{1}{2}\Sigma\Big(\frac{d\xi}{dx}p_{xx} + \frac{d\eta}{dy}p_{yy} + \frac{d\zeta}{dz}p_{zz} + \Big(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\Big)p_{yz} + \Big(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\Big)_{zx}$$

$$+ \Big(\frac{d\xi}{dy} + \frac{d\eta}{dx}\Big)p_{xy}\Big)dV,$$

the summation being extended to the whole contents of the sphere.

We find, as we ought, that these quantities have the same value, namely

$$U = -\tfrac{2}{3}\pi a^5 me(e+2f). \quad . \quad . \quad . \quad . \quad (98)$$

We may now suppose that the tangential action on the surface arises from a layer of particles in contact with it, the particles

[redacted]

whose density is ρ, and having its normal inclined θ to the axes of z; then the tangential force upon it will be

$$\rho R \delta S \sin\theta = 2T\delta S, \quad \ldots \ldots \quad (99)$$

T being, as before, the tangential force on each side of the surface. Putting $\rho = \frac{1}{2\pi}$ as in equation (34)*, we find

$$R = -2\pi m a (e + 2f). \quad \ldots \ldots \quad (100)$$

The displacement of electricity due to the distortion of the sphere is

$$\Sigma \delta S \tfrac{1}{2} \rho t \sin\theta \text{ taken over the whole surface;} \quad . \quad (101)$$

and if h is the electric displacement per unit of volume, we shall have

$$\tfrac{4}{3}\pi a^3 h = \tfrac{2}{3} a^4 e, \quad \ldots \ldots \quad (102)$$

or

$$h = \frac{1}{2\pi} a e; \quad \ldots \ldots \quad (103)$$

so that

$$R = 4\pi^2 m \frac{e + 2f}{e} h, \quad \ldots \ldots \quad (104)$$

or we may write

$$R = -4\pi E^2 h, \quad \ldots \ldots \quad (105)$$

provided we assume

$$E^2 = -\pi m \frac{e + 2f}{e}. \quad \ldots \ldots \quad (106)$$

Finding e and f from (87) and (90), we get

$$E^2 = \pi m \frac{3}{1 + \frac{5}{3}\frac{m}{\mu}}. \quad \ldots \ldots \quad (107)$$

The ratio of m to μ varies in different substances; but in a medium whose elasticity depends entirely upon forces acting between pairs of particles, this ratio is that of 6 to 5, and in this case

$$E^2 = \pi m. \quad \ldots \ldots \quad (108)$$

When the resistance to compression is infinitely greater than the resistance to distortion, as in a liquid rendered slightly elastic by gum or jelly, a hypothetically "perfect" solid*, in which

$$5m = 6\mu, \quad \ldots \ldots \quad (110)$$

so that we must use equation (108).

Prop. XIV.—To correct the equations (9) † of electric currents for the effect due to the elasticity of the medium.

We have seen that electromotive force and electric displacement are connected by equation (105). Differentiating this equation with respect to t, we find

$$\frac{dR}{dt} = -4\pi E^2 \frac{dh}{dt}, \quad \ldots \ldots \quad (111)$$

showing that when the electromotive force varies, the electric [redacted]

equivalent to a current, and this current must be taken into account in equations (9) and added to r. The three equations then become [redacted]

$$p = \frac{1}{4\pi}\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz} - \frac{1}{E^2}\frac{dP}{dt}\right),$$

[redacted]

ments is

$$U = -\Sigma\tfrac{1}{2}(Pf + Qg + Rh)\delta V, \quad . \quad . \quad . \quad . \quad (116)$$

where P, Q, R are the forces, and f, g, h the displacements. Now when there is no motion of the bodies or alteration of forces, it appears from equations (77)* that

$$P = -\frac{d\Psi}{dx}, \quad Q = -\frac{d\Psi}{dy}, \quad R = -\frac{d\Psi}{dz}; \quad . \quad (118)$$

and we know by (105) that

$$P = -4\pi E^2 f, \quad Q = -4\pi E^2 g, \quad R = -4\pi E^2 h; \quad . \quad (119)$$

whence

$$U = \frac{1}{8\pi E^2}\Sigma\left(\overline{\frac{d\Psi}{dx}}\Big|^2 + \overline{\frac{d\Psi}{dy}}\Big|^2 + \overline{\frac{d\Psi}{dz}}\Big|^2\right)\delta V. \quad . \quad (120)$$

Integrating by parts throughout all space, and remembering that Ψ vanishes at an infinite distance,

$$U = -\frac{1}{8\pi E^2}\Sigma\Psi\left(\frac{d^2\Psi}{dx^2} + \frac{d^2\Psi}{dy^2} + \frac{d^2\Psi}{dz^2}\right)\delta V\,; \quad (121)$$

[illegible]

or by (115),

$$U = \tfrac{1}{2}\Sigma(\Psi e)\delta V. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122)$$

Now let there be two electrified bodies, and let e_1 be the distri-

[illegible]

or

$$F\,dr = c_1 \frac{d\Psi_2}{dr} dr, \quad . \quad . \quad . \quad . \quad . \quad . \quad (126)$$

where F is the resistance and dr the motion.

If the body e_2 be small, then if r is the distance from e_2, equation (123) gives

$$\Psi_2 = E^2 \frac{e_2}{r};$$

whence

$$F = -E^2 \frac{e_1 e_2}{r^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

or the force is a repulsion varying inversely as the square of the distance.

Now let η_1 and η_2 be the same quantities of electricity measured statically, then we know by definition of electrical quantity

$$F = -\frac{\eta_1 \eta_2}{r^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (128)$$

and this will be satisfied provided

$$\eta_1 = E e_1 \text{ and } \eta_2 = E e_2; \quad . \quad . \quad . \quad . \quad (129)$$

so that the quantity E previously determined in Prop. XIII. is the number by which the electrodynamic measure of any quantity of electricity must be multiplied to obtain its electrostatic measure.

That electric current which, circulating round a ring whose area is unity, produces the same effect on a distant magnet as a

[redacted]

[redacted]

[redacted]

Prop. XVI.—To find the rate of propagation of transverse vibrations through the elastic medium of which the cells are

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

on each surface; then the capacity .

$$C=\frac{e}{\Psi_1-\Psi_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (138)$$

Within the dielectric we have the variation of Ψ perpendicular to the surface

$$=\frac{\Psi_1-\Psi_2}{\theta}.$$

Beyond either surface this variation is zero.

[redacted]

$$\frac{\Psi_1-\Psi_2}{4\pi E^2\theta}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

and we deduce the whole capacity of the apparatus,

$$C=\frac{S}{\;} \qquad (140)$$

axis, depends upon the velocity of light whose vibrations are parallel to that axis, or whose plane of polarization is perpendicular to that axis.

In a uniaxal crystal, the axial value of E will depend on the velocity of the extraordinary ray, and the equatorial value will depend on that of the ordinary ray.

In "positive" crystals, the axial value of E will be the least and in negative the greatest.

The value of D, which varies inversely as E^2, will, *cæteris*

XIV. *On Physical Lines of Force.* *By* J. C. Maxwell, *F.R.S., Professor of Natural Philosophy in King's College, London*.*

Part IV.—*The Theory of Molecular Vortices applied to the Action of Magnetism on Polarized Light.*

THE connexion between the distribution of lines of magnetic conceives magnetism to consist in currents of a fluid whose direcare then produced in the substance by the action of a magnet or of an electric current, the plane of polarization of the transmitted the vortices accounts for induced currents, is supported by the opinion of Professor W. Thomson*. In fact the whole theory of

[redacted]

[redacted]

A and B have been fully investigated by M. Verdet*, who has shown that the rotation is strictly proportional to the thickness [redacted]

to the magnetizing force, the rotation is as the cosine of that inclination. D has been supposed to give the true relation between the rotation of different rays; but it is probable that C must be taken into account in an accurate statement of the phenomena. The rotation varies, not exactly inversely as the square of the wave-length, but a little faster; so that for the highly refrangible rays the rotation is greater than that given by this law, but more nearly as the index of refraction divided by the square of the [redacted]

wave-length.

The relation (E) between the amount of rotation and the size of the vortices shows that different substances may differ in rotaand diamagnetism, and does not require us to admit either M. Weber's theory of the mutual action of electric particles in motion, or our theory of cells and cell-walls.

in the manner of its action as well as in the intensity of [illegible] ts magand the energy

$$E = \tfrac{1}{2}\Sigma dm r^2 \omega^2 = \tfrac{1}{2}A\omega,$$

$$= \frac{1}{8\pi}\mu\alpha^2 V \text{ by Prop. VI.*},$$

whence

$$A = \frac{1}{4\pi}\mu r\alpha V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (144)$$

for the axis of x, with similar expressions for the other axes, V being the volume, and r the radius of the vortex.

Prop. XIX.—To determine the conditions of undulatory movaried motion is

$$\left.\begin{aligned} \mathrm{Y} &= k_2 \frac{dy}{dz} - \frac{1}{4\pi}\mu r \frac{d\alpha}{dt}. \\ \text{Similarly,} \qquad \mathrm{X} &= k_1 \frac{dx}{dz} + \frac{1}{4\pi}\mu r \frac{d\beta}{dt}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (145)$$

The whole force acting upon a stratum whose thickness is dz and area unity is $\frac{d\mathrm{X}}{dz}\,dz$ in the direction of x, and $\frac{d\mathrm{Y}}{dz}\,dz$ in direction of y. The mass of the stratum is ρdz, so that we have as the equations of motion,

$$\left.\begin{aligned} \rho \frac{d^2x}{dt^2} &= \frac{d\mathrm{X}}{dz} = k_1 \frac{d^2x}{dz^2} + \frac{d}{dz}\frac{1}{4\pi}\mu r \frac{d\beta}{dt}, \\ \rho \frac{d^2y}{dt^2} &= \frac{d\mathrm{Y}}{dz} = k_2 \frac{d^2y}{dz^2} - \frac{d}{dz}\frac{1}{4\pi}\mu r \frac{d\alpha}{dt}. \end{aligned}\right\} \quad . \quad (146)$$

Now the changes of velocity $\frac{d\alpha}{dt}$ and $\frac{d\beta}{dt}$ are produced by the motion of the medium containing the vortices, which distorts and twists every element of its mass; so that we must refer to Prop. X.* to determine these quantities in terms of the motion. We find there at equation (68),

$$\delta\alpha = \alpha \frac{d}{dx}\delta x + \beta \frac{d}{dy}\delta x + \gamma \frac{d}{dz}\delta x \quad . \ . \ . \ (68).$$

Since δx and δy are functions of z and t only, we may write this equation

$$\left.\begin{aligned} \frac{d\alpha}{dt} &= \gamma \frac{d^2x}{dz\,dt}; \\ \text{and in like manner,} \qquad \frac{d\beta}{dt} &= \gamma \frac{d^2y}{dz\,dt}; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (147)$$

so that if we now put $k_1 = a^2\rho$, $k_2 = b^2\rho$, and $\frac{1}{4\pi}\frac{\mu r}{\rho}\gamma = c^2$, we may write the equations of motion

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= a^2 \frac{d^2x}{dz^2} + c^2 \frac{d^3y}{dz^2\,dt}, \\ \frac{d^2y}{dt^2} &= b^2 \frac{d^2y}{dz^2} - c^2 \frac{d^3x}{dz^2\,dt}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (148)$$

These equations may be satisfied by the values

$$x = \mathrm{A}\cos(nt - mz + \alpha), \qquad . \quad . \quad . \quad . \quad . \quad (149)$$

provided

$$\left.\begin{array}{l}(n^2-m^2a^2)\mathrm{A}=m^2nc^2\mathrm{B},\\ (n^2-m^2b^2)\mathrm{B}=m^2nc^2\mathrm{A}.\end{array}\right\} \quad . \quad . \quad . \quad . \quad (150)$$

and

Multiplying the last two equations together, we find

$$(n^2-m^2a^2)(n^2-m^2b^2)=m^4n^2c^4, \quad . \quad . \quad . \quad (151)$$

an equation quadratic with respect to m^2, the solution of which is

$$m^2=\frac{2n^2}{a^2+b^2\mp\sqrt{(a^2-b^2)^2+4n^2c^4}}. \quad . \quad . \quad (152)$$

These values of m^2 being put in the equations (150) will each give a ratio of A and B.

$$\frac{\mathrm{A}}{\mathrm{B}}=\frac{a^2-b^2\mp\sqrt{(a^2-b^2)^2+4n^2c^4}}{2nc^2},$$

which being substituted in equations (149), will satisfy the original equations (148). The most general undulation of such a medium is therefore compounded of two elliptic undulations of different eccentricities travelling with different velocities and rovibration whose periodic time is $\frac{2\pi}{n}$, and wave-length $\frac{2\pi}{p}=\lambda$, propagated in the direction of z with a velocity $\frac{n}{p}=v$, while the plane of the vibration revolves about the axis of z in the positive direction so as to complete a revolution when $z=\frac{2\pi}{q}$.

Now let us suppose c^2 small, then we may write

$$p=\frac{n}{a} \text{ and } q=\frac{n^2c^2}{2a^3}; \quad . \quad . \quad . \quad . \quad . \quad (157)$$

and remembering that $c^2=\frac{1}{4\pi}\frac{r}{\rho}\mu\gamma$, we find

$$q=\frac{\pi}{2}\frac{r}{\rho}\frac{\mu\gamma}{\lambda^2 v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (158)$$

Here r is the radius of the vortices, an unknown quantity. ρ is the density of the luminiferous medium in the body, which is also unknown; but if we adopt the theory of Fresnel, and make s the density in space devoid of gross matter, then

$$\rho=si^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

where i is the index of refraction.

On the theory of MacCullagh and Neumann,

$$\rho=s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (160)$$

in all bodies.

μ is the coefficient of magnetic induction, which is unity in empty space or in air.

γ is the velocity of the vortices at their circumference estimated in the ordinary units. Its value is unknown, but it is of refraction of that ray in the body,

$$\lambda = \frac{\Lambda}{i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (162)$$

Also v, the velocity of light in the substance, is related to V, the

[redacted]

Hence if z be the thickness of the substance through which the ray passes, the angle through which the plane of polarization will be turned will be in degrees,

[redacted]