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# Bivariate Gončarov polynomials and integer sequences

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**Abstract** Univariate Gončarov polynomials arose from the Gončarov interpolation problem in numerical analysis. They provide a natural basis of polynomials for working with $\boldsymbol{u}$-parking functions, which are integer sequences whose order statistics are bounded by a given sequence $\boldsymbol{u}$. In this paper, we study multivariate Gončarov polynomials, which form a basis of solutions for multivariate Gončarov interpolation problem. We present algebraic and analytic properties of multivariate Gončarov polynomials and establish a combinatorial relation with integer sequences. Explicitly, we prove that multivariate Gončarov polynomials enumerate $k$-tuples of integers sequences whose order statistics are bounded by certain weights along lattice paths in $\mathbb{N}^k$. It leads to a higher-dimensional generalization of parking functions, for which many enumerative results can be derived from the theory of multivariate Gončarov polynomials.

**Keywords** Gončarov polynomials, interpolation, parking functions, order statistics

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## 1 Introduction

Gončarov polynomials arose in a special case of Hermite interpolation in numerical analysis.

**Gončarov interpolation.** Given two sequences of real or complex numbers $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$, find a polynomial $p(x)$ of degree $n$ such that for each $i, 0 \leqslant i \leqslant n$, the $i$-th derivative $p^{(i)}(x)$ evaluated at $a_i$ equals $b_i$.

The natural basis of polynomials for this interpolation problem is the sequence of Gončarov polynomials (see [1, 2, 9, 13]), which are polynomials $g_n(x; a_0, a_1, \ldots, a_{n-1})$ defined by the biorthogonality relation:

$$\varepsilon(a_i)D^i g_n(x; a_0, a_1, \ldots, a_{n-1}) = n!\delta_{in},$$

where $D$ is the differential operator, and $\varepsilon(a)$ is evaluation at $a$. A special case of this is *Abel interpolation,* where the point $a_i$ is the integer $i$. The Gončarov polynomials for this case are the Abel polynomials, which frequently appear in enumerative combinatorics and are closely related to counting of labeled trees and some special integer sequences called parking functions.

For a sequence of numbers $(x_1, x_2, \ldots, x_n)$, the order statistics are the sequence $x_{(1)} \leqslant x_{(2)} \leqslant \cdots \leqslant x_{(n)}$ obtained by rearranging the original sequence in non-decreasing order. An ordinary parking function is

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a sequence $(x_1, \dots, x_n)$ of non-negative integers whose order statistics satisfy $x_{(i)} < i$ for all $i$. Ordinary parking functions originated in the theory of hashing and searching in computer science (see [7, 8]), and have been studied extensively in combinatorial literature. In particular, the number of ordinary parking functions is an evaluation of an Abel polynomial.

There are many generalizations of ordinary parking functions. One of them is a set of integer sequences called $\boldsymbol{u}$-parking functions, which are studied in [10, 11, 14, 16–19]. Let $\boldsymbol{u} = (u_1, u_2, \dots)$ be a sequence of non-decreasing positive integers. A $\boldsymbol{u}$-parking function of length $n$ is a sequence $(x_1, x_2, \dots, x_n)$ of non-negative integers whose order statistics satisfy $x_{(i)} < u_i$. Gončarov polynomials form a basis of polynomials for working with $\boldsymbol{u}$-parking functions. For example, the number of $\boldsymbol{u}$-parking functions of length $n$ is $(-1)^n g_n(0; u_1, u_2, \dots, u_n)$. When the sequence $\boldsymbol{u}$ is an arithmetic progression, the Gončarov polynomial is an Abel polynomial. Gončarov polynomials have many nice algebraic and analytic properties, which extend automatically to $\boldsymbol{u}$-parking functions. In particular, Gončarov polynomials satisfy a linear recursion obtained by expanding $x^n$ as a linear combination of Gončarov polynomials, which leads to a decomposition of an arbitrary sequence of non-negative integers into two subsequences: a maximum $\boldsymbol{u}$-parking function and a subsequence consisting of terms of higher values. Many enumerative results of $\boldsymbol{u}$-parking functions can be derived from this decomposition.

Gončarov interpolation problem has a natural generalization in multi-variables, whose solutions are characterized by multivariate Gončarov polynomials. Explicitly, let $n_1, n_2, \dots, n_k$ be positive integers. Given a set of nodes $S = \{z_{i_1,\dots,i_k} \in \mathbb{R}^k \mid 0 \leqslant i_j \leqslant n_j\}$ and values $\{b_{i_1,\dots,i_k} \in \mathbb{R} \mid 0 \leqslant i_j \leqslant n_j\}$, we want to find a polynomial $P \in \mathbb{R}[x_1, \dots, x_k]$ whose partial derivatives satisfy

$$\frac{\partial^{i_1+\cdots+i_k}}{\partial x_1^{i_1} \cdots \partial x_k^{i_k}} P(z_{i_1,\dots,i_k}) = b_{i_1,\dots,i_k}.$$

Such a problem has a unique solution in the space of polynomials of coordinate degree $(n_1, \dots, n_k)$, i.e., in the space $\mathcal{P}_{n_1,\dots,n_k}$, where

$$\mathcal{P}_{n_1,\dots,n_k} = \left\{ P \in \mathbb{R}[x_1, \dots, x_k] \,\middle|\, P = \sum_{0 \leqslant i_j \leqslant n_j} a_{i_1,\dots,i_k} x_1^{i_1} x_2^{i_2} \cdot x_k^{i_k},\ a_{i_1,\dots,a_k} \in \mathbb{R} \right\}.$$

Multivariate Gončarov polynomials are the natural basis of polynomials for this interpolation problem. In this paper, we establish algebraic and analytic properties of multivariate Gončarov polynomials analogous to those of the univariate ones, including a linear recurrence, generating functions, differential and integral relations, a shift invariance formula, a perturbation formula, and a Sheffer relation. We show that the coefficients of multivariate Gončarov polynomials can be expressed combinatorially in terms of ordered partitions, which are partitions of a finite set whose blocks are linearly ordered. More interestingly, multivariate Gončarov polynomials have a close relationship with order statistics of integer sequences. The main combinatorial result of this paper is that a Gončarov polynomial in $k$ variables with index $(n_1, n_2, \dots, n_k)$ and the node set $S$ enumerates the set of $k$-tuples of integer sequences $(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_k)$, where $\boldsymbol{x}_i$ is a sequence of length $n_i$, and the order statistics of these sequences are bounded by certain weights along some lattice paths from the origin to $(n_1, \dots, n_k)$ in $\mathbb{Z}_+^k$. The weights on the lattice paths are given by the node set $S$. (See Section 6 for the exact description.) This leads to a new, higher-dimensional generalization of classical parking functions, for which multivariate Gončarov polynomials are the corresponding algebraic counterpart. Any reasonable formula for multivariate Gončarov polynomials yields automatically a reasonable formula for higher-dimensional parking functions.

We remark that another type of Gončarov polynomial has been investigated by He [3]. Given $n$ nodes in $\mathbb{R}^k$, He interpolated partial derivatives there with polynomials of degree $n - 1$ in $k$ variables. Since there are more polynomial coefficients than the number of interpolation conditions, additional constraints are added in order to make the interpolation unique, much in the same way as in Kergin interpolation [5]. The multivariate Gončarov polynomials discussed in this paper are different from He's.

For simplicity and clarity we present our results in two variables only. It is easy to extend all the results to polynomials with $k$ variables, for any positive integer $k$. The paper is organized as follows.

First we review the theory of sequences of biorthogonal polynomials and results of univariate Gončarov polynomials in Section 2. Then we introduce the bivariate generalization of Gončarov polynomials and describe their algebraic and analytic properties in Section 3. In Section 4 we present a combinatorial description of the coefficients of bivariate Gončarov polynomials in terms of ordered partitions. The last two sections are focused on the relations between bivariate Gončarov polynomials $g_{m,n}((x,y);Z)$ and order statistics of integer sequences. Section 5 contains explicit formulas for the case $n=1$, where the Gončarov polynomials can also be expressed in terms of $\boldsymbol{u}$-parking functions. The combinatorial representation for general cases is given in Section 6, where we propose a notion of 2-dimensional parking functions, establish their relation with bivariate Gončarov polynomials, and derive a formula for the sum enumerator of 2-dimensional parking functions.

## 2 Sequences of biorthogonal polynomials and univariate Gončarov polynomials

We begin by giving an outline of the theory of sequences of polynomials biorthogonal to a sequence of linear functionals. The details can be found in [10].

Let $\mathcal{P}$ be the vector space of all polynomials in the variable $x$ over $\mathbb{R}$. Let $D:\mathcal{P}\to\mathcal{P}$ be the differentiation operator, and $\varepsilon(a):\mathcal{P}\to\mathbb{R}$ be the evaluation at $a\in\mathbb{R}$.

Let $\varphi_s(D), s=0,1,2,\ldots$ be a sequence of linear operators on $\mathcal{P}$ of the form

$$\varphi_s(D)=D^s\sum_{r=0}^{\infty}b_{sr}D^r, \tag{2.1}$$

where the coefficients $b_{s0}$ are assumed to be non-zero. There exists a unique sequence $p_n(x), n=0,1,2,\ldots$ of polynomials such that $p_n(x)$ has degree $n$ and

$$\varepsilon(0)\varphi_s(D)p_n(x)=n!\delta_{sn}, \tag{2.2}$$

where $\delta_{sn}$ is the Kronecker delta.

The polynomial sequence $p_n(x)$ is said to be *biorthogonal* to the sequence $\varphi_s(D)$ of operators, or, the sequence $\varepsilon(0)\varphi_s(D)$ of linear functionals. Using Cramer's rule to solve the linear system and Laplace's expansion to group the results, we can express $p_n(x)$ by the the following *determinantal formula*:

$$p_n(x)=\frac{n!}{b_{00}b_{10}\cdots b_{n0}}\begin{vmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0,n-1} & b_{0n}\\ 0 & b_{10} & b_{11} & \cdots & b_{1,n-2} & b_{1,n-1}\\ 0 & 0 & b_{20} & \cdots & b_{2,n-3} & b_{2,n-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & b_{n-1,0} & b_{n-1,1}\\ 1 & x & x^2/2! & \cdots & x^{n-1}/(n-1)! & x^n/n! \end{vmatrix}. \tag{2.3}$$

The series $\{p_n(x)\}_{n=0}^{\infty}$ forms a basis of $\mathcal{P}$. A special example of sequences of biorthogonal polynomials is the Gončarov polynomials. Let $(a_0,a_1,a_2,\ldots)$ be a sequence of numbers or variables called *nodes.* The sequence of *Gončarov polynomials* $g_n(x;a_0,a_1,\ldots,a_{n-1}),\ n=0,1,2,\ldots$ is the sequence of polynomials biorthogonal to the operators

$$\varphi_s(D)=D^s\sum_{r=0}^{\infty}\frac{a_s^rD^r}{r!}=\varepsilon(a_s)D^s.$$

As indicated by the notation, $g_n(x;a_0,a_1,\ldots,a_{n-1})$ depends only on the nodes $a_0,a_1,\ldots,a_{n-1}$. Indeed, one can write down a determinantal formula of $g_n(x;a_0,a_1,\ldots,a_{n-1})$ using Equation (2.3). In particular, when all the $a_i$ equal $a$, we have $g_n(x;a,a,\ldots,a)=(x-a)^n$ and Gončarov interpolation is just expansion as a power series at $x = a$. When $a_0, a_1, a_2, \ldots$ form an arithmetic progression $a, a+b, a+2b, \ldots$, we get Abel polynomials $g_n(x; a, a+b, a+2b, \ldots, a+(n-1)b) = (x-a)(x-a-nb)^{n-1}$.

Gončarov polynomials have many nice algebraic and analytic properties, which make them very useful in analysis and combinatorics. Here we list some basic properties whose proofs can be found in [10].

1. *Expansion formula.* If $p(x)$ is a polynomial of degree $n$, then

$$p(x) = \sum_{i=0}^{n} \frac{\varepsilon(a_i) D^i p(x)}{i!} g_i(x; a_0, a_1, \ldots, a_{i-1}).$$

2. *Linear recurrence.* Let $p(x) = x^n$ in the expansion formula, we have

$$x^n = \sum_{i=0}^{n} \binom{n}{i} a_i^{n-i} g_i(x; a_0, a_1, \ldots, a_{i-1}).$$

3. *Appell relation.*

$$\mathrm{e}^{xt} = \sum_{n=0}^{\infty} g_n(x; a_0, a_1, \ldots, a_{n-1}) \frac{t^n \mathrm{e}^{a_n t}}{n!}.$$

4. *Differential relations.* The Gončarov polynomials can be equivalently defined by the differential relations $Dg_n(x; a_0, a_1, \ldots, a_{n-1}) = ng_{n-1}(x; a_1, a_2, \ldots, a_{n-1})$, with initial conditions $g_n(a_0; a_0, a_1, \ldots, a_{n-1}) = \delta_{0n}$.

5. *Integral relations.*

$$g_n(x; a_0, a_1, \ldots, a_{n-1}) = n \int_{a_0}^{x} g_{n-1}(t; a_1, a_2, \ldots, a_{n-1}) dt = n! \int_{a_0}^{x} dt_1 \int_{a_1}^{t_1} dt_2 \cdots \int_{a_{n-1}}^{t_{n-1}} dt_n.$$

6. *Shift invariance.* $g_n(x+\xi; a_0+\xi, a_1+\xi, \ldots, a_{n-1}+\xi) = g_n(x; a_0, a_1, \ldots, a_{n-1})$.

7. *Perturbation formula.*

$$\begin{aligned} & g_n(x; a_0, \ldots, a_{m-1}, a_m + b_m, a_{m+1}, \ldots, a_{n-1}) \\ & \quad = g_n(x; a_0, \ldots a_{m-1}, a_m, a_{m+1}, \ldots, a_{n-1}) \\ & \qquad - \binom{n}{m} g_{n-m}(a_m + b_m; a_m, a_{m+1}, \ldots, a_{n-1}) g_m(x; a_0, a_1, \ldots, a_{m-1}). \end{aligned}$$

8. *Sheffer relation.*

$$g_n(x+y; a_0, \ldots, a_{n-1}) = \sum_{i=0}^{n} \binom{n}{i} g_{n-i}(y; a_i, \ldots, a_{n-1}) x^i.$$

In particular,

$$g_n(x; a_0, \ldots, a_{n-1}) = \sum_{i=0}^{n} \binom{n}{i} g_{n-i}(0, a_i, \ldots, a_{n-1}) x^i.$$

That is, coefficients of Gončarov polynomials are constant terms of (shifted) Gončarov polynomials.

9. *Combinatorial representation.* Let $\boldsymbol{u} = (u_1, u_2, \ldots)$ be a sequence of non-decreasing positive integers. Recall that a $\boldsymbol{u}$-parking function of length $n$ is a sequence $(x_1, x_2, \ldots, x_n)$ of non-negative integers whose order statistics satisfy $x_{(i)} < u_i$ for $1 \leqslant i \leqslant n$. Denote by $\mathcal{PK}_n(\boldsymbol{u})$ the set of $\boldsymbol{u}$-parking functions of length $n$, and by $PK_n(\boldsymbol{u})$ the size of $\mathcal{PK}_n(\boldsymbol{u})$. Then we have

$$\begin{aligned} PK_n(\boldsymbol{u}) = PK_n(u_1, u_2, \ldots, u_n) &= g_n(x; x-u_1, x-u_2, \ldots, x-u_n) \\ &= g_n(0; -u_1, -u_2, \ldots, -u_n) \\ &= (-1)^n g_n(0; u_1, u_2, \ldots, u_n). \end{aligned}$$

For more properties and computations of parking functions via Gončarov polynomials, please refer to [10–12]. In particular, the sum enumerator and factorial moments of the sums are computed. For

$\boldsymbol{u}$-parking functions, the sum enumerator is a specialization of $g_n(x; a_0, a_1, \ldots, a_{n-1})$ with $a_i = 1 + q + \cdots + q^{u_i - 1}$. Generating functions for factorial moments of sums of $\boldsymbol{u}$-parking functions are given in [10], while the explicit formulas for the first and second factorial moments of sums of $\boldsymbol{u}$-parking functions are given in [11], and in [12] for all factorial moments for classical parking functions where the sequence $u_i$ is an arithmetic progression.

## 3 Algebraic properties of bivariate Gončarov polynomials

Sequences of multivariate biorthogonal polynomials can be defined by replacing the differential operator $D$ with partial derivatives. It turns out that we can define the multivariate Gončarov polynomials which are the basis of solutions of the general Gončarov interpolation problem. Many algebraic and analytic properties listed in the preceding section can be extended to the multivariate generalization. For simplicity and clarity, we state and prove the results for the bivariate case only. It is routine to extend the results to more variables.

In this paper $\mathbb{N} = \{0, 1, 2, \ldots\}$ represents the set of all natural numbers.

**Definition 1.** A finite subset $S$ of $\mathbb{N}^2$ is called a *lower set* if for any $(m, n) \in S$, we have that $(i, j) \in S$ for any $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$.

Let $\mathcal{P}_S$ be the space of bivariate polynomials

$$\mathcal{P}_S = \left\{ P \,\middle|\, P(x, y) = \sum_{(i,j) \in S} a_{i,j} x^i y^j,\ a_{i,j} \in \mathbb{R} \right\}. \tag{3.1}$$

Some special cases of the lower set $S$ are

1. $S = [0, 1, \ldots, m] \times [0, 1, \ldots, n]$. Then $\mathcal{P}_S$ is the space of bivariate polynomials of coordinate degree $(m, n)$. It is denoted by $\Pi^2_{m,n}$.

2. $S = \{(i, j) \mid 0 \leqslant i + j \leqslant n\}$. Then $\mathcal{P}_S$ is the space of bivariate polynomials of (total) degree $n$. It is denoted by $\Pi^2_n$.

**Bivariate Gončarov interpolation problem.** Given a lower set $S$, a set of nodes $\{z_{i,j} = (x_{i,j}, y_{i,j}) \mid (i, j) \in S\}$ and a set of real numbers $\{b_{i,j} \mid (i, j) \in S\}$, find a polynomial $P \in \mathcal{P}_S$ satisfying

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} P(x_{i,j}, y_{i,j}) = b_{i,j} \tag{3.2}$$

for all $(i, j) \in S$.

That the bivariate Gončarov interpolation problem is uniquely solvable follows from the fact that with the appropriate ordering of the monomials $x^i y^j$ and the corresponding order of the functionals $\varepsilon((x_{i,j}, y_{i,j}))\partial^{i+j}/\partial x^i \partial y^j$, the matrix of the linear system for the coefficients of the solution is an upper diagonal matrix with non-zero entries on the diagonal. Here $\varepsilon((x_0, y_0))$ is evaluation at $(x_0, y_0)$. An appropriate order for both monomials and functionals is the faster scan order, i.e., $(i, j) < (i', j')$ if $i + j < i' + j'$, or $i + j = i' + j'$ and $i < i'$.

In particular, given $(i, j) \in S$, there exist unique polynomials $P_{i,j} \in \mathcal{P}_S$ satisfying

$$\frac{\partial^{r+s}}{\partial x^r \partial y^s} P_{i,j}(x_{r,s}, y_{r,s}) = \delta_{r,i}\delta_{s,j} \tag{3.3}$$

for all $(r, s) \in S$. The uniqueness of the solution implies that

- $P_{i,j}$ is a linear combination of only those monomials $x^k y^\ell$ with $k \leqslant i$ and $\ell \leqslant j$, i.e., $P_{i,j} \in \Pi^2_{i,j}$.
- $P_{i,j}$ does not depend on $S$ as long as $(i, j) \in S$.
- $P_{i,j}$ does not depend on the node $z_{i,j} = (x_{i,j}, y_{i,j})$.

In light of the above, we have

**Definition 2.** Given a set of nodes $Z = \{z_{i,j} = (x_{i,j}, y_{i,j}) \mid (i,j) \in \mathbb{N}^2\}$, the bivariate Gončarov polynomial $g_{i,j}((x,y);Z)$ is the unique polynomial in $\Pi^2_{i,j}$ satisfying

$$\frac{\partial^{r+s}}{\partial x^r \partial y^s} g_{i,j}(z_{r,s}) = i!j!\delta_{r,i}\,\delta_{s,j} \tag{3.4}$$

for all $0 \leqslant r \leqslant i$ and $0 \leqslant s \leqslant j$.

Note that $g_{i,j}((x,y);Z)$ depends only on the nodes $\{z_{r,s} \mid 0 \leqslant r \leqslant i,\ 0 \leqslant s \leqslant j,\ (r,s) \neq (i,j)\}$. Nevertheless, for consistency we use the same (infinite) set $Z$ throughout the paper.

**Example 3.1.** Some examples of bivariate Gončarov polynomials.

1. Comparing the definitions of univariate and bivariate Gončarov polynomials, we have

$$g_{i,0}((x,y);Z) = g_i(x; x_{0,0}, x_{1,0}, \ldots, x_{i-1,0}),$$
$$g_{0,j}((x,y);Z) = g_j(y; y_{0,0}, y_{0,1}, \ldots, y_{0,j-1}),$$

where $g_i$ and $g_j$ are univariate Gončarov polynomials.

2. When the set of node $Z$ is a grid, i.e., $(x_{i,j}, y_{i,j}) = (\alpha_i, \beta_j)$ for some sequences $\{\alpha_i\}$ and $\{\beta_j\}$, then

$$g_{i,j}((x,y);Z) = g_i(x; \alpha_0, \ldots, \alpha_{i-1})\ g_j(y; \beta_0, \ldots, \beta_{j-1})$$

is the product of univariate Gončarov polynomials in variables $x$ and $y$. In particular, when $\alpha_i = \alpha$ and $\beta_j = \beta$ are constants, we have $g_{i,j}((x,y);Z) = (x-\alpha)^i(y-\beta)^j$ and the Gončarov interpolation is just the Taylor expansion at $(x,y) = (\alpha,\beta)$. When $x_i$ and $y_j$ are arithmetic progressions, $g_{i,j}((x,y);Z)$ is an product of Abel polynomials in $x$ and $y$ respectively. Multivariate Gončarov polynomials associated to a grid were studied in [4].

3. Using MATLAB, we obtain

$$g_{11}((x,y);Z) = xy - y_{10}x - x_{01}y + y_{10}x_{00} + x_{01}y_{00} - x_{00}y_{00},$$

and

$$\begin{aligned} g_{21}((x,y);Z) = {} & x^2y - 2x_{11}xy + (2x_{01}x_{11} - x_{01}^2)y - y_{20}x^2 \\ & + (2x_{11}y_{10} + 2y_{20}x_{10} - 2x_{10}y_{10})x - x_{00}^2y_{00} + 2x_{11}x_{00}y_{00} \\ & - 2x_{01}x_{11}y_{00} + x_{01}^2y_{00} + y_{20}x_{00}^2 + 2x_{10}y_{10}x_{00} - 2x_{11}y_{10}x_{00} - 2y_{20}x_{10}x_{00}. \end{aligned}$$

Many properties of the univariate Gončarov polynomials can be extended to the multivariable generalizations.

**Theorem 3.2** (Expansion formula)**.** *For any* $p \in \Pi^2_{m,n}$,

$$p(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{1}{i!j!}\left[\varepsilon(z_{i,j})\frac{\partial^{i+j}}{\partial x^i \partial y^j}p(x,y)\right] g_{i,j}((x,y);Z). \tag{3.5}$$

*Proof.* This follows immediately from the biorthogonality of the functionals and the definition of bivariate Gončarov polynomials. □

For $p(x,y) = x^my^n$, we obtain the linear recursion

**Theorem 3.3** (Linear recursion)**.**

$$x^my^n = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}x_{i,j}^{m-i}y_{i,j}^{n-j}g_{i,j}((x,y);Z). \tag{3.6}$$

**Theorem 3.4** (Appell relation)**.**

$$\mathrm{e}^{sx+ty} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g_{m,n}((x,y);Z)\frac{s^m\mathrm{e}^{x_{m,n}s}}{m!}\frac{t^n\mathrm{e}^{y_{m,n}t}}{n!}. \tag{3.7}$$

*Proof.* We use, for any $u_{m,i}$,

$$\sum_{m=0}^{\infty}\sum_{i=0}^{m}u_{m,i}=\sum_{i=0}^{\infty}\sum_{m=i}^{\infty}u_{m,i}\quad\text{and}\quad\sum_{m=i}^{\infty}\frac{1}{m!}\binom{m}{i}x^{m-i}s^{m}=\frac{s^i}{i!}\sum_{r=0}^{\infty}\frac{1}{r!}x^r s^r=\frac{s^i}{i!}\mathrm{e}^{xs}$$

to obtain

$$\begin{aligned}\mathrm{e}^{sx+ty}&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^m s^m}{m!}\frac{y^n t^n}{n!}\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{1}{m!}\binom{m}{i}\frac{1}{n!}\binom{n}{j}x_{i,j}^{m-i}s^m y_{i,j}^{n-j}t^n g_{i,j}((x,y);Z)\\&=\sum_{i=0}^{\infty}\sum_{m=i}^{\infty}\sum_{j=0}^{\infty}\sum_{n=j}^{\infty}\frac{1}{m!}\binom{m}{i}\frac{1}{n!}\binom{n}{j}x_{i,j}^{m-i}s^m y_{i,j}^{n-j}t^n g_{i,j}((x,y);Z)\\&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}g_{i,j}((x,y);Z)\sum_{m=i}^{\infty}\frac{1}{m!}\binom{m}{i}x_{i,j}^{m-i}s^m\sum_{n=j}^{\infty}\frac{1}{n!}\binom{n}{j}y_{i,j}^{n-j}t^n\\&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}g_{i,j}((x,y);Z)\frac{s^i}{i!}\frac{t^j}{j!}\mathrm{e}^{x_{m,n}s+y_{m,n}t}.\end{aligned}$$

The proof is complete. □

Next, we derive the differential and integral relations for bivariate Gončarov polynomials. For the set $Z=\{z_{i,j}\mid i,j\in\mathbb{N}\}$, let $LZ=\{w_{i,j}\mid w_{i,j}=z_{i+1,j},\ i,j\in\mathbb{N}\}$, i.e., the first column of the set of nodes is deleted and the others are moved to the left. Similarly define the set $DZ$ as $\{w_{i,j}\mid w_{i,j}=z_{i,j+1},\ i,j\in\mathbb{N}\}$, in which the first row of nodes of $Z$ is deleted and the others are moved down. Then we have

**Theorem 3.5** (Differential relations)**.**

$$\frac{\partial}{\partial x}g_{m,n}((x,y);Z)=mg_{m-1,n}((x,y);LZ),\tag{3.8}$$

$$\frac{\partial}{\partial y}g_{m,n}((x,y);Z)=ng_{m,n-1}((x,y);DZ).\tag{3.9}$$

*Proof.* We prove the first relation only. Let $h(x,y)=\frac{\partial}{\partial x}g_{m,n}((x,y);Z)$. Then $h$ is a polynomial in $\Pi^2_{m-1,n}$. From the definition of $g_{m,n}((x,y);Z)$, $h$ satisfies the interpolation conditions

$$\frac{\partial^{r+s}}{\partial x^r\partial y^s}h(z_{r+1,s})=\frac{\partial^{r+1+s}}{\partial x^{r+1}\partial y^s}g_{m,n}(z_{r+1,s})=0$$

for $0\leqslant r\leqslant m-1$, $0\leqslant s\leqslant n$ and $(r,s)\neq(m-1,n)$.

For $(r,s)=(m-1,n)$,

$$\frac{\partial^{m-1+n}}{\partial x^{m-1}\partial y^n}h(z_{m,n})=m!n!=m(m-1)!n!,$$

which are the same conditions that $mg_{m-1,n}((x,y);LZ)$ satisfies. Since the interpolation is unique, $h(x,y)=mg_{m-1,n}((x,y);LZ)$. □

**Corollary 3.6.** *The general differential formula is*

$$\frac{\partial^{i+j}}{\partial x^i\partial y^j}g_{m,n}((x,y);Z)=(m)_i(n)_j\,g_{m-i,n-j}((x,y);D^jL^iZ),\tag{3.10}$$

*where for any number $r$, $(r)_k=r(r-1)\cdots(r-k+1)$ is the $k$-th lower factorial of $r$.* □

**Theorem 3.7** (Integral relations)**.** *Let $m\geqslant 1$ and*

$$G_{m,n}((x,y);LZ)=m\int^{x}g_{m-1,n}((t,y);LZ)dt,$$

*where* $\int^x g_{m-1,n}((t,y);LZ)dt$ *is that indefinite integral with respect to* $x$ *of* $g_{m-1,n}((x,y);LZ_{m,n})$ *which contains no constant term, i.e., no term of the form* $h(y)$.

*Also, let* $P_n(y)$ *be the solution of the univariate Gončarov interpolation problem*

$$\varepsilon(y_{0,j})\frac{d^j}{dy^j}P_n(y)=\varepsilon(x_{0,j},y_{0,j})\frac{\partial^j}{\partial y^j}G_{m,n}((x,y);LZ)$$

*for* $j=0,\dots,n$. *Then*

$$g_{m,n}((x,y);Z)=G_{m,n}((x,y);LZ)-P_n(y). \tag{3.11}$$

*Similarly, if* $n\geqslant 1$ *and*

$$H_{m,n}((x,y);LZ)=n\int^y g_{m,n-1}((x,s);DZ)\,ds$$

*and* $Q_m(x)$ *is the solution of the univariate Gončarov interpolation problem*

$$\varepsilon(x_{i,0})\frac{d^i}{dx^i}Q_m(x)=\varepsilon(x_{i,0},y_{i,0})\frac{\partial^i}{\partial x^i}H_{m,n}((x,y);DZ)$$

*for* $i=0,\dots,m$, *then*

$$g_{m,n}((x,y);Z)=H_{m,n}((x,y);DZ)-Q_m(x). \tag{3.12}$$

*Proof.* Again we only prove Formula (3.11). From the differential relations we have

$$\frac{\partial}{\partial x}g_{m,n}((x,y);Z)=mg_{m-1,n}((x,y);LZ).$$

From the definition of $G_{m,n}((x,y);LZ)$,

$$\frac{\partial}{\partial x}G_{m,n}((x,y);LZ)=mg_{m-1,n}((x,y);LZ).$$

Thus the difference $G_{m,n}((x,y);LZ_{m,n})-g_{m,n}((x,y);Z_{m,n})$ is a function of $y$, actually a polynomial $R_n(y)$ of degree at most $n$ in $y$. For $R_n(y)$,

$$\varepsilon(y_{0,j})\frac{d^j}{dy^j}R_n(y)=\varepsilon(x_{0,j},y_{0,j})\frac{\partial^j}{\partial y^j}G_{m,n}((x,y);LZ).$$

From the differential relation, the interpolation conditions are satisfied for all nodes $(x_{ij},y_{ij})$ with $i\geqslant 1$. Subtracting $R_n$ forces the interpolation conditions to be satisfied for $i=0$. □

**Theorem 3.8** (Shift invariance). *Let* $E^a$ *be the shift operator* $(E^af)(x)=f(x-a)$. *Since shifts of a function commute with derivatives and since* $(E^af)(x+a)=f(x)$, *we have a shift invariance formula*

$$g_{m,n}((x+\xi,y+\eta);Z+(\xi,\eta))=g_{m,n}((x,y);Z). \tag{3.13}$$

*In particular,*

$$g_{m,n}((0,0);Z-(x,y))=g_{m,n}((x,y);Z), \tag{3.14}$$

*where* $Z-(x,y)=\{(x_{i,j}-x,y_{i,j}-y)\mid i,j\in\mathbb{N}\}$.

**Theorem 3.9** (Perturbation formula). *Let us perturb the* $(i_0,j_0)$*-th node of* $Z$ *to be* $z^*_{i_0,j_0}$ *and denote the new set of nodes by* $Z^*$. *Then for* $(i_0,j_0)\leqslant(m,n)$,

$$g_{m,n}((x,y);Z^*)=g_{m,n}((x,y);Z)-\binom{m}{i_0}\binom{n}{j_0}g_{m-i_0,n-j_0}(z^*_{i_0,j_0};D^{j_0}L^{i_0}Z)g_{i_0,j_0}((x,y);Z).$$

*Proof.* Note that $g_{m,n}((x,y);Z)$ satisfies the same interpolation conditions as $g_{m,n}((x,y);Z^*)$ except at the $(i_0,j_0)$-th node, where

$$\frac{\partial^{i_0+j_0}}{\partial x^{i_0}\partial y^{j_0}}g_{m,n}(z^*_{i_0,j_0};Z)$$

might not be 0. Subtracting the polynomial

$$\frac{\partial^{i_0+j_0}}{\partial x^{i_0}\partial y^{j_0}} g_{m,n}(z^*_{i_0,j_0};Z)\frac{1}{i_0!}\frac{1}{j_0} g_{i_0,j_0}((x,y);Z)$$

from $g_{m,n}((x,y);Z)$ yields a polynomial satisfying all of the conditions that $g_{m,n}((x,y);Z^*)$ does. Using the differential relations we obtain the perturbation formula. □

**Theorem 3.10** (Sheffer relation)**.**

$$g_{m,n}((x+b,y+c);Z)=\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j} g_{m-i,n-j}((b,c);L^iD^jZ)x^iy^j. \tag{3.15}$$

*Proof.* Using the Taylor expansion about $(b,c)$ and the differential relations

$$\begin{aligned} g_{m,n}((x+b,y+c);Z) &= \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{1}{i!\,j!}\frac{\partial^{i+j}}{\partial x^i\partial y^j}[g_{m,n}((b,c);Z)]x^iy^j \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{1}{i!\,j!}\cdot(m)_i(n)_j\ g_{m-i,n-j}((b,c);L^iD^jZ)x^iy^j \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j} g_{m-i,n-j}((b,c);L^iD^jZ)x^iy^j, \end{aligned}$$

we finish the proof. □

## 4 Coefficients of bivariate Gončarov polynomials

In this section we give an explicit formula and a combinatorial interpretation of the coefficients of bivariate Gončarov polynomials. First we show that it suffices to consider only the constant terms.

Setting $b=c=0$ in the Sheffer relation (3.15), we get

$$g_{m,n}((x,y);Z)=\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j} g_{m-i,n-j}((0,0);L^iD^jZ)x^iy^j. \tag{4.1}$$

Hence the coefficients of bivariate Gončarov polynomials are products of binomial coefficients and the constant terms of (shifted) bivariate Gončarov polynomials.

**Example 4.1.** Some explicit formulas for $g_{m,n}((0,0);Z)$.

$$\begin{aligned} g_{1,1}((0,0);Z) &= x_{0,0}y_{1,0}+x_{0,1}y_{0,0}-x_{0,0}y_{0,0} \\ g_{2,1}((0,0);Z) &= -x_{0,0}^2y_{0,0}+2x_{1,1}x_{0,0}y_{0,0}-2x_{0,1}x_{1,1}y_{0,0}+x_{0,1}^2y_{,00} \\ &\quad + y_{2,0}x_{0,0}^2+2x_{1,0}y_{1,0}x_{0,0}-2x_{1,1}y_{1,0}x_{0,0}-2y_{2,0}x_{1,0}x_{0,0}. \end{aligned}$$

Let $f_{m,n}$ be the number of monomials in the constant term of $g_{m,n}((x,y);Z)$, where we ignore the sign and count multiplicity. For example, $f_{1,1}=3$ and $f_{2,1}=13$. The initial values of $f_{m,n}$ are listed in the following table, where the rows are indexed by $m$ and columns are indexed by $n$.

| $(m,n)$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 3 | 13 | 75 | 541 |
| 1 | 1 | 3 | 13 | 75 | 541 | |
| 2 | 3 | 13 | 75 | 541 | | |
| 3 | 13 | 75 | 541 | | | |
| 4 | 75 | 541 | | | | |
| 5 | 541 | | | | | |

We observe that $f_{m,n}$ depends only on $m+n$. The sequence $1, 1, 3, 13, 75, 541, \ldots$ is the sequence A000670 in the On-line Encyclopedia of Integer Sequences, which counts the number of preferential arrangements, or ordered partitions of a set. Let $S$ be a set with $n$ elements. An ordered partition of $S$ is an ordered list $(B_1, B_2, \ldots, B_k)$ of disjoint non-empty subsets of $S$ such that $B_1 \cup B_2 \cup \cdots \cup B_k = S$. Write an ordered partition by $B_1/B_2/\cdots/B_k$. Let $\alpha_n$ be the number of ordered partitions of an $n$-element set $S$. For example, $\alpha_3 = 13$, where the ordered partitions of $\{1, 2, 3\}$ are 123, 12/3, 13/2, 23/1, 1/23, 2/13, 3/12, 1/2/3, 1/3/2, 2/1/3, 2/3/1, 3/1/2 and 3/2/1. The exponential generating function of $\alpha_n$ is well-known, for example, see [15, 3.15.10],

$$\sum_{n\geqslant 0} \alpha_n \frac{x^n}{n!} = \frac{1}{2-\mathrm{e}^x}.$$

Given an ordered partition $\pi = B_1/B_2/\cdots/B_k$ of $S$, for two elements $i, j \in S$, we say that $i$ is of a lower rank than $j$ if $i \in B_s$, $j \in B_t$ and $s < t$.

Let

$$A_{m,n} = \{a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n\}$$

be a set consisting of $m+n$ elements. For an ordered partition $\pi$ of $A_{m,n}$ and any element $c \in A_{m,n}$, let

$$\sigma(c) = \#\{a_i \in A_{m,n} \mid a_i \text{ has a lower rank than } c\}, \tag{4.2}$$

$$\rho(c) = \#\{b_j \in A_{m,n} \mid b_j \text{ has a lower rank than } c\}. \tag{4.3}$$

We have the following combinatorial interpretation of $g_{m,n}((0,0); Z)$.

**Theorem 4.2.**

$$g_{m,n}((0,0); Z_{m,n}) = \sum_{\pi} (-1)^{|\pi|} \prod_{a_i \in A_{m,n}} x_{\sigma(a_i),\rho(a_i)} \prod_{b_j \in A_{m,n}} y_{\sigma(b_j),\rho(b_j)}, \tag{4.4}$$

*where $\pi$ ranges over all ordered partitions of $A_{m,n}$, and $|\pi|$ is the number of blocks in $\pi$. As a corollary, $f_{m,n} = \alpha_{m+n}$.*

*Proof.* Denote by $\Gamma(m,n)$ the sum on the right-hand side of Equation (4.4). By convention, set $\Gamma(0,0) = 1$. Equation (4.4) holds for $(m,0)$, $(0,n)$, which is proved in [10, Theorem 4.2].

In the linear recurrence (3.6), letting $x = y = 0$ and moving $g_{m,n}$ to the left-hand side of the equation, we obtain

$$g_{m,n}((0,0); Z) = -\sum_{(i,j)\neq(m,n)} \binom{m}{i}\binom{n}{j} x_{i,j}^{m-i} y_{i,j}^{n-j} g_{i,j}((0,0); Z). \tag{4.5}$$

We shall show that $\Gamma(m,n)$ satisfies the same recurrence relation. Let $OP(i,j)$ be the set of all the ordered partitions of the set $A_{m,n}$ where the last block $B_k$ contains $m-i$ elements from $\{a_1, \ldots, a_m\}$ and $n-j$ elements from $\{b_1, \ldots, b_n\}$. Note that $(i,j) \neq (m,n)$ since $B_k$ cannot be empty. There are $\binom{m}{i}\binom{n}{j}$ ways to choose elements of $B_k$. Let $T = A_{m,n} \setminus B_k$. Hence the contribution of ordered partitions in $OP(i,j)$ to $\Gamma(m,n)$ can be computed as

$$(-1) \cdot \binom{m}{i}\binom{n}{j} x_{i,j}^{m-i}\ y_{i,j}^{n-j} \cdot \sum_{\pi'} (-1)^{|\pi'|} \prod_{a_s \in T} x_{\sigma(a_s),\rho(a_s)} \prod_{b_t \in T} y_{\sigma(b_t),\rho(b_t)},$$

where $\pi'$ ranges over all ordered partitions of $T$. Summing over all $i, j$ with $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$ and $(i,j) \neq (m,n)$, and using an inductive argument on $m+n$, one proves that $g_{m,n}((0,0); Z) = \Gamma(m,n)$. □

**Example 4.3.** The following table shows the correspondence between ordered partitions of the set $\{a_1, a_2, b\}$ and the monomials in $g_{2,1}((0,0); Z)$.

| Ordered partition | Term in $g_{2,1}((0,0);Z)$ |
|---|---|
| $a_1a_2b$ | $-x_{0,0}^2y_{0,0}$ |
| $a_1a_2/b$ | $x_{0,0}^2y_{2,0}$ |
| $a_1/a_2b$, $a_2/a_1b$ | $2x_{0,0}x_{1,0}y_{1,0}$ |
| $a_1b/a_2$, $a_2b/a_1$ | $2x_{0,0}y_{0,0}x_{1,1}$ |
| $b/a_1a_2$ | $y_{0,0}x_{0,1}^2$ |
| $a_1/a_2/b$, $a_2/a_1/b$ | $-2x_{0,0}x_{1,0}y_{2,0}$ |
| $a_1/b/a_2$, $a_2/b/a_1$ | $-2x_{0,0}y_{1,0}x_{1,1}$ |
| $b/a_1/a_2$, $b/a_2/a_1$ | $-2y_{0,0}x_{0,1}x_{1,1}$ |

**Corollary 4.4.** *The bivariate Gončarov polynomial $g_{m,n}((x,y);Z)$ is homogeneous of degree $(m,n)$, i.e., viewed as a polynomial of $x,y,x_{i,j}$ and $y_{i,j}$, every term in the expansion of $g_{m,n}((x,y);Z)$ has total $x$-degree $m$ and $y$-degree $n$.*

This follows from Theorem 4.2 and Equation (4.1) given at the beginning of this section.

## 5 Polynomial $g_{m,1}((x,y);Z)$ and integer sequences

The rest of the paper is devoted to the combinatorics theory of bivariate Gončarov polynomials. In the case that $Z$ is a grid, i.e., $(x_{i,j},y_{i,j})=(\alpha_i,\beta_j)$ for some sequences $\{\alpha_i\}$ and $\{\beta_j\}$, we have that $g_{m,n}((x,y),Z)=g_m(x;\alpha_0,\dots,\alpha_{m-1})g_n(y;\beta_0,\dots,\beta_{n-1})$, which counts the number of pairs of sequences $(\boldsymbol{a},\boldsymbol{b})$, where $\boldsymbol{a}=(a_0,a_1,\dots,a_{m-1})$ satisfies $0\leqslant a_{(i)}<x-\alpha_i$, and $\boldsymbol{b}=(b_0,b_1,\dots,b_{n-1})$ satisfies $0\leqslant b_{(j)}<y-\beta_j$, for $0\leqslant i<m$, $0\leqslant j<n$.

When $Z$ is not a grid, $g_{m,n}((x,y);Z)$ also counts pairs of sequences with certain constraints on their order statistics. In this section we describe the combinatorial representation of $g_{m,1}((x,y);Z)$ and express $g_{m,1}((x,y);Z)$ in terms of $\boldsymbol{u}$-parking functions. The combinatorial interpretation for general $g_{m,n}((x,y);Z)$ will be given in the next section.

**Notation.** Let $Z_{m,n}\subset Z$ be the set of nodes $Z_{m,n}=\{z_{i,j}=(x_{i,j},y_{i,j})\mid 0\leqslant i\leqslant m,\ 0\leqslant j\leqslant n\}$. Assume that $x_{i,j},y_{i,j}$ are all positive integers, where $x_{i,j}\geqslant x_{i',j'}$ and $y_{i,j}\geqslant y_{i',j'}$ whenever $i\leqslant i'\leqslant m$ and $j\leqslant j'\leqslant n$. Let $x>x_{0,0}$ and $y>y_{0,0}$ be two sufficiently large positive integers.

Set $u_{i,j}=x-x_{i,j}$ and $v_{i,j}=y-y_{i,j}$. Hence $u_{i,j}\leqslant u_{i',j'}$ and $v_{i,j}\leqslant v_{i',j'}$ whenever $i\leqslant i'\leqslant m$ and $j\leqslant j'\leqslant n$.

Let $\mathcal{S}(m,n)$ be the set of pairs of integer sequences $(\boldsymbol{a},\boldsymbol{b})$, where $\boldsymbol{a}=(a_0,a_1,\dots,a_{m-1})$ is of length $m$ whose terms satisfy $0\leqslant a_i<x$, and $\boldsymbol{b}=(b_0,b_1,\dots,b_{n-1})$ is of lengths $n$ whose terms satisfy $0\leqslant b_j<y$. Clearly $|\mathcal{S}(m,n)|=x^my^n$.

First we look at the combinatorial description of $g_{m,1}((x,y);Z)$ when $m\leqslant 2$. We would start with the linear recurrence

$$x^my^n=\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}x_{i,j}^{m-i}y_{i,j}^{n-j}g_{i,j}((x,y);Z). \tag{5.1}$$

**Example 5.1.** The case $m=n=1$. Equation (5.1) becomes

$$xy=x_{0,0}y_{0,0}g_{0,0}((x,y);Z)+y_{1,0}g_{1,0}((x,y),Z)+x_{0,1}g_{0,1}((x,y);Z)+g_{1,1}((x,y);Z). \tag{5.2}$$

The left-hand side of Equation (5.2) counts the set $\mathcal{S}(1,1)$, i.e., pairs of integers $(a,b)$, where $0\leqslant a<x$ and $0\leqslant b<y$. The right-hand side has four terms:

1. $x_{0,0}y_{0,0}g_{0,0}=x_{0,0}y_{0,0}$. This counts the pairs such that $a\geqslant u_{0,0}$ and $b\geqslant v_{0,0}$.
2. $y_{1,0}g_{1,0}((x,y),Z)$. This counts the pairs such that $a<u_{0,0}$ but $b\geqslant v_{1,0}$.
3. $x_{0,1}g_{0,1}((x,y);Z)$. This counts the pairs such that $b<v_{0,0}$ but $a\geqslant u_{0,1}$.

4. $g_{1,1}((x,y);Z)$: Comparing the above three cases with $\mathcal{S}(1,1)$, we obtain that this term counts the number of pairs $(a,b)$ in the union of the two sets:

(1) $a < u_{0,0}$ and $b < v_{1,0}$, or (2) $a < u_{0,1}$ and $b < v_{0,0}$.

**Example 5.2.** The case $m = 2$ and $n = 1$. Equation (5.1) becomes

$$x^2 y = \sum_{i=0}^{2} \binom{2}{i} x_{i,0}^{2-i} y_{i,0} g_{i,0}((x,y),Z) + \sum_{i=0}^{2} \binom{2}{i} x_{i,1}^{2-i} g_{i,1}((x,y),Z). \tag{5.3}$$

The left-hand side of (5.3) counts number of pairs in $\mathcal{S}(2,1)$. The right-hand side has six terms:

1. $y_{0,0}x_{0,0}^2 g_{0,0} = y_{0,0}x_{0,0}^2$, which counts the pairs such that $a_{(0)} \geqslant u_{0,0}$, $a_{(1)} \geqslant u_{0,0}$, and $b \geqslant v_{0,0}$.
2. The term $2y_{1,0}x_{1,0}g_{1,0}$ counts the pairs such that $a_{(0)} < u_{0,0}$ but $a_{(1)} \geqslant u_{1,0}$, and $b \geqslant v_{1,0}$.
3. The term $y_{2,0}g_{2,0}$ counts the pairs such that $a_{(0)} < u_{0,0}$ and $a_{(1)} < u_{1,0}$, and $b \geqslant v_{2,0}$.
4. The term $x_{0,1}^2 g_{0,1}$ counts the pairs such that $a_{(0)} \geqslant u_{0,1}$, $a_{(1)} \geqslant u_{0,1}$, and $b < v_{0,0}$.
5. The term $2x_{1,1}g_{1,1}$ counts the union of pairs such that (1) $a_{(0)} < u_{0,0}$ but $a_{(1)} \geqslant u_{1,1}$, and $b < v_{1,0}$; or (2) $a_{(0)} < u_{0,1}$ but $a_{(1)} \geqslant u_{1,1}$, and $b < v_{0,0}$.
6. The last term is $g_{2,1}((x,y);Z)$. Combining the above, we derive that this term counts the number of pairs in the union of the following three sets:

(a) $a_{(0)} < u_{0,0}$, $a_{(1)} < u_{1,0}$ and $b < v_{2,0}$.

(b) $a_{(0)} < u_{0,0}$, $a_{(1)} < u_{1,1}$ and $b < v_{1,0}$.

(c) $a_{(0)} < u_{0,1}$, $a_{(1)} < u_{1,1}$ and $b < v_{0,0}$.

For general $m$, we have the following description for $g_{m,1}((x,y);Z)$.

**Theorem 5.3.** *The bivariate Gončarov polynomial $g_{m,1}((x,y);Z)$ counts the number of pairs of non-negative integer sequences $((a_0,a_1,\ldots,a_{n-1}),b)$ which are in the union*

$$\bigcup_{i=0}^{m} \{(\boldsymbol{a},b) : b < v_{i,0},\ \boldsymbol{a} \in \mathcal{PK}_m(u_{0,0},\ldots,u_{i-1,0},u_{i,1},\ldots,u_{m-1,1})\}, \tag{5.4}$$

*where $u_{i,j} = x - x_{i,j}$ and $v_{i,j} = y - y_{i,j}$. Equivalently, the set* (5.4) *can be expressed as the union of disjoint sets $A_i$, where*

$$A_i = \{(\boldsymbol{a},b) \mid v_{i-1,0} \leqslant b < v_{i,0},\ \boldsymbol{a} \in \mathcal{PK}_m(u_{0,0},\ldots,u_{i-1,0},u_{i,1},\ldots,u_{m-1,1})\} \tag{5.5}$$

*for $i = 0,1,\ldots,m$. Here by convention $u_{-1,0} = v_{-1,0} = 0$.*

*Proof.* We prove by induction on $m$. For $m = 1,2$, Theorem 5.3 has been confirmed in Examples 5.1 and 5.2. Assume that the theorem holds for all positive integers less than $m$. We shall prove that it holds for $m$ as well.

Let $\mathcal{S}_i = \{((a_0,a_1,\ldots,a_{m-1}),b) \mid 0 \leqslant a_i < x,\ v_{i-1,0} \leqslant b < v_{i,0}\}$ for $i = 0,1,\ldots,m+1$, where by convention $v_{-1,0} = 0$ and $v_{m+1} = y$. Let $g_{m,1}((xy);Z) \cap \mathcal{S}_i$ be the set of sequences $((a_0,\ldots,a_{m-1}),b)$ in $\mathcal{S}_i$ that are counted by $g_{m,1}((x,y);Z)$. We just need to show that $g_{m,1}((x,y);Z) \cap \mathcal{S}_i = A_i$ for $0 \leqslant i \leqslant m$, where $A_i$ is given in (5.5), and $g_{m,1}((x,y);Z) \cap \mathcal{S}_{m+1} = \emptyset$.

For any sequence $\boldsymbol{a} = (a_0,a_1,\ldots,a_{m-1})$, compare its order statistics with $t_0,\ldots,t_{m-1}$, where

$$t_j = \begin{cases} u_{j,0}, & \text{if } j < i, \\ u_{j,1}, & \text{if } j \geqslant i. \end{cases}$$

Let $\kappa = \kappa(\boldsymbol{a})$ be the maximum index such that

$$a_{(l)} < t_l, \quad \text{for } l = 0,1,\ldots,\kappa-1. \tag{5.6}$$

It follows that $a_{(\kappa)} \geqslant t_\kappa$. Then the sequence $\boldsymbol{a}$ can be decomposed into two subsequences, the first containing $\kappa$ smallest terms and belonging to $\mathcal{PK}_\kappa(t_0,\ldots,t_{\kappa-1})$, and the second containing $m-\kappa$ largest terms, each of which is in $[t_\kappa, x)$. Hence the set $\mathcal{S}_i$ is a disjoint union of $m+1$ subsets $\mathcal{S}_i(k)$, where

$$\mathcal{S}_i(k) = \{(\boldsymbol{a};b) \mid \kappa(\boldsymbol{a}) = k,\ v_{i-1,0} \leqslant b < v_{i,0}\ \}, \qquad i = 0,1,\ldots,m.$$

The recurrence (5.1) with $n = 1$ becomes

$$x^m y = \sum_{i=0}^{m} \binom{m}{i} x_{i,0}^{m-i} y_{i,0} g_{i,0}((x,y),Z) + \sum_{i=0}^{m} \binom{m}{i} x_{i,1}^{m-i} g_{i,1}((x,y),Z). \tag{5.7}$$

We analyze the contribution of each term on the right-hand side of Equation (5.7) to the set $\mathcal{S}_i$. First,

$$g_{j,0}((x,y),Z) = PK_j(u_{0,0},\ldots,u_{j-1,0}).$$

Hence the term

$$\binom{m}{j} x_{j,0}^{m-j} y_{j,0} g_{j,0}((x,y),Z_{i,0}) \tag{5.8}$$

counts the pairs of sequences $((a_0,a_1,\ldots,a_{m-1}),b)$ such that

1. $v_{j,0} \leqslant b < y$;
2. the first $j$ order statistics of the sequence $\boldsymbol{a}$ form a subsequence in $\mathcal{PK}_j(u_{0,0},\ldots,u_{j-1,0})$;
3. the largest $m-j$ terms of the sequence $\boldsymbol{a}$ are in $[u_{j,0},x)$.

Comparing with the set $\mathcal{S}_i$, we notice that the term (5.8) counts the number of all sequences in the set $\mathcal{S}_i(j)$ if $j < i$. While $j \geqslant i$, it does not count any sequence in $\mathcal{S}_i$. In particular, the sum of terms (5.8) from $j = 0,\ldots,m$ already counts all the sequences in $\mathcal{S}_{m+1}$.

Next, we look at the term

$$\binom{m}{j} x_{j,1}^{m-j} g_{j,1}((x,y),Z) \tag{5.9}$$

for $j < m$. By inductive hypothesis, (5.9) counts the pairs of sequences $(\boldsymbol{a},b) = ((a_0,\ldots,a_n),b)$ such that

1. The largest $m-j$ terms of the sequence $\boldsymbol{a}$ are in $[u_{j,1},x)$.
2. Let $\boldsymbol{a}'$ be the subsequence of $\boldsymbol{a}$ consisting of the smallest $j$ terms. Then $(\boldsymbol{a}',b)$ belongs to the disjoint union of

$$\bigcup_{k=0}^{j} \{(\boldsymbol{a}';b) \mid v_{k-1,0} \leqslant b < v_{k,0},\ \boldsymbol{a}' \in \mathcal{PK}_j(u_{0,0},\ldots,u_{k-1,0},u_{k,1},\ldots,u_{j-1,1})\}.$$

Comparing with the set $\mathcal{S}_i$, we notice that when $j < i$, (5.9) counts nothing in $\mathcal{S}_i$; when $i \leqslant j < m$, it counts all sequences in $\mathcal{S}_i(j)$.

Excluding those sequences counted by (5.8) and (5.9) from the set $\mathcal{S}_i$, we obtain that $g_{m,1}((x,y);Z) \cap \mathcal{S}_i = \mathcal{S}_i(m)$, which is exactly the set $A_i$ defined in (5.5). □

Since $v_{i,0} - v_{i-1,0} = y_{i-1,0} - y_{i,0}$, from (5.5) we get the following relation between $g_{m,1}((x,y);Z_{m,1})$ and the univariate Gončarov polynomials $g_m(x)$.

**Corollary 5.4.**

$$g_{m,1}((x,y);Z) = \sum_{i=0}^{m} (y_{i-1,0} - y_{i,0}) g_m(x;x_{0,0},\ldots,x_{i-1,0},x_{i,1},\ldots,x_{m-1,1}),$$

*where* $y_{-1,0} = y$.

Using the shift invariance formula for univariate Gončarov polynomials and the combinatorial representation by $\boldsymbol{u}$-parking functions, we obtain the following corollary.

**Corollary 5.5.**

$$(-1)^m g_{m,1}((0,0);Z) = \sum_{i=0}^{m} (y_{i-1,0} - y_{i,0}) PK_m(x_{0,0},\ldots,x_{i-1,0},x_{i,1},\ldots,x_{m-1,1}),$$

*where* $y_{-1,0} = 0$.

## 6 Polynomial $g_{m,n}((x,y);Z)$ and 2-dimensional parking function

For general $m,n$ the bivariate Gončarov polynomial $g_{m,n}((x,y);Z)$ can also be interpreted as the number of pairs of sequences $(\boldsymbol{a},\boldsymbol{b})$ in $\mathcal{S}(m,n)$ whose order statistics satisfy certain constraints. In this section we give the description for general cases, which leads to a definition of 2-dimensional parking functions.

We adopt the notation defined in the previous section. In addition, let $R_{m,n}$ be the directed graph whose vertices are all the lattice points $\{(i,j): 0 \leqslant i \leqslant m,\ 0 \leqslant j \leqslant n\}$, and whose edges are all the North and East unit steps connecting the vertices, where the North step is $N=(0,1)$ and the East step is $E=(1,0)$. Associate to each edge $e$ of $R_{m,n}$ a weight $wt(e)$ by letting

$$wt(e) = \begin{cases} u_{i,j}, & \text{if } e \text{ is an } E\text{-step from } (i,j) \text{ to } (i+1,j), \\ v_{i,j}, & \text{if } e \text{ is an } N\text{-step from } (i,j) \text{ to } (i,j+1). \end{cases} \tag{6.1}$$

For a lattice path $P$ from the origin $O=(0,0)$ to the point $A=(m,n)$ in $R_{m,n}$ consisting of $N$- and $E$-steps, record the steps of $P$ as $P = p_1p_2\cdots p_{m+n}$, where $p_i \in \{N,E\}$ and there are exactly $m$ $E$-steps and $n$ $N$-steps. Given a pair of sequences $(\boldsymbol{a},\boldsymbol{b}) \in \mathcal{S}(m,n)$, we say that the order statistics of $(\boldsymbol{a},\boldsymbol{b})$ are bounded by the path $P$ with respect to the set $\boldsymbol{U} = \{(u_{i,j},v_{i,j}) : 0 \leqslant i \leqslant m,\ 0 \leqslant j \leqslant n\}$ if and only if the order statistics of $\boldsymbol{a}$ are bounded by the weight on the $E$-steps, and the order statistics of $\boldsymbol{b}$ are bounded by the weight on the $N$-steps of $P$. In other words, for $r = 1,2,\ldots,m+n$,

$$\begin{cases} a_{(i)} < u_{i,j}, & \text{if } p_r \text{ is an } E\text{-step from } (i,j) \text{ to } (i+1,j), \\ b_{(j)} < v_{i,j}, & \text{if } p_r \text{ is an } N\text{-step from } (i,j) \text{ to } (i,j+1). \end{cases} \tag{6.2}$$

Denote by $\mathcal{S}_{m,n}(P;\boldsymbol{U})$ the subset of $\mathcal{S}(m,n)$ consisting of the pairs of sequences $(\boldsymbol{a},\boldsymbol{b})$ whose order statistics are bounded by $P$ with respect to $\boldsymbol{U}$. Our main result is the following theorem.

**Theorem 6.1.** *The bivariate Gončarov polynomial $g_{m,n}((x,y);Z)$ counts the number of pairs of sequences in $\mathcal{S}(m,n)$ whose order statistics are bounded by some lattice path from $O$ to $A=(m,n)$. That is, $g_{m,n}((x,y);Z)$ is the cardinality of the union*

$$\bigcup_{P:O\to A} \mathcal{S}_{m,n}(P;\boldsymbol{U}),$$

*where $P$ ranges over all lattice paths from $O$ to $A$ using $N$- and $E$-steps only, and the set $\boldsymbol{U} = \{(u_{i,j},v_{i,j}) : 0 \leqslant i \leqslant m,\ 0 \leqslant j \leqslant n\}$ is determined by $Z$ by the relations $u_{i,j} = x - x_{i,j}$, $v_{i,j} = y - y_{i,j}$.*

We explain Theorem 6.1 using the example with $m = n = 2$.

**Example 6.2.** Similar to the examples in the previous subsection, one can compute by hand that $g_{2,2}((x,y);Z)$ counts the number of pairs of sequences $((a_0,a_1);(b_0,b_1))$ in the union of the six sets described by the conditions listed below:

| Subset | $a_{(0)}$ | $a_{(1)}$ | $b_{(0)}$ | $b_{(1)}$ |
|---|---|---|---|---|
| Subset 1 | $< u_{0,0}$ | $< u_{1,0}$ | $< v_{2,0}$ | $< v_{2,1}$ |
| Subset 2 | $< u_{0,0}$ | $< u_{1,1}$ | $< v_{1,0}$ | $< v_{2,1}$ |
| Subset 3 | $< u_{0,0}$ | $< u_{1,2}$ | $< v_{1,0}$ | $< v_{1,1}$ |
| Subset 4 | $< u_{0,1}$ | $< u_{1,1}$ | $< v_{0,0}$ | $< v_{2,1}$ |
| Subset 5 | $< u_{0,1}$ | $< u_{1,2}$ | $< v_{0,0}$ | $< v_{1,1}$ |
| Subset 6 | $< u_{0,2}$ | $< u_{1,2}$ | $< v_{0,0}$ | $< v_{0,1}$ |

The graph $D_{2,2}$ is shown in Figure 1, where the label on each edge is the corresponding bound of the order statistic.

There are six lattice paths from $O$ to $A$ in $D_{2,2}$, as illustrated in Figure 2. The $i$-th subset listed in the above table is exactly $\mathcal{S}_{2,2}(P_i;\boldsymbol{U})$ where $P_i$ is the $i$-th lattice path in Figure 2.

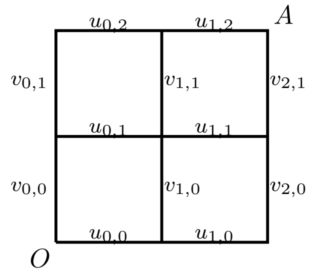


Figure 1 The graph $D_{2,2}$

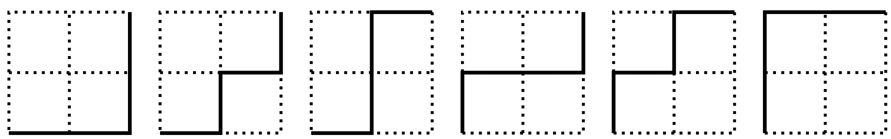

Figure 2 Lattice paths in $D_{2,2}$

Again if $u_{i,j} = u_i$ and $v_{i,j} = v_j$, i.e., $Z_{2,2}$ is a grid, then all six sets are identical, which is

$$\{((a_0, a_1); (b_0, b_1)) : a_{(0)} < u_0, a_{(1)} < u_1; b_{(0)} < v_0, b_{(1)} < v_1\},$$

and is counted by the product of univariate Gončarov polynomials $g_2(x; x_{0,0}, x_{1,0})g_2(y; y_{0,0}, y_{0,1})$. □

*Proof of Theorem* 6.1. For any pair of sequences $\boldsymbol{c} = (\boldsymbol{a}, \boldsymbol{b}) \in \mathcal{S}(m, n)$, construct a subgraph $G(\boldsymbol{c})$ of $D_{m,n}$ as follows:

- $O = (0, 0)$ is a vertex of $G(\boldsymbol{c})$.
- For any vertex $(i, j)$ of $G(\boldsymbol{c})$,
  - If $a_{(i)} < u_{i,j}$, then add the vertex $(i + 1, j)$ and the $E$-step $\{(i, j), (i + 1, j)\}$ to $G(\boldsymbol{c})$.
  - If $b_{(j)} < v_{i,j}$, then add the vertex $(i, j + 1)$ and the $N$-step $\{(i, j), (i, j + 1)\}$ to $G(\boldsymbol{c})$.

Clearly $G(\boldsymbol{c})$ is a connected graph containing the origin.

**Lemma 6.3.** *If the edges* $e_1 = \{(i, j), (i + 1, j)\}$ *and* $e_2 = \{(i, j), (i, j + 1)\}$ *are both in* $G(\boldsymbol{c})$*, then edges* $e_3 = \{(i + 1, j), (i + 1, j + 1)\}$ *and* $e_4 = \{(i, j + 1), (i + 1, j + 1)\}$ *are also in* $G(\boldsymbol{c})$*.*

*Proof.* The edge $e_1$ is in $G(\boldsymbol{c})$ means that the lattice point $(i + 1, j)$ is in $G(\boldsymbol{c})$ and $a_{(i)} < u_{i,j}$. The edge $e_2$ is in $G(\boldsymbol{c})$ means that the lattice point $(i, j + 1)$ is in $G(\boldsymbol{c})$ and $b_{(j)} < v_{i,j}$. Since $u_{i,j} \leqslant u_{i,j+1}$ and $v_{i,j} \leqslant v_{i+1,j}$, we have $a_{(i)} < u_{i,j+1}$ and $b_{(j)} < v_{i+1,j}$. By the definition of $G(\boldsymbol{c})$, the lattice point $(i + 1, j + 1)$ and edges $e_3$ and $e_4$ are in $G(\boldsymbol{c})$. □

Order the lattice points of $\mathbb{Z}^2$ by letting $(i, j) \preceq (i', j')$ if and only if $i \leqslant i'$ and $j \leqslant j'$.

**Lemma 6.4.** *The set of vertices of* $G(\boldsymbol{c})$ *has a unique maximal vertex under the order* $\preceq$*.*

*Proof.* First the vertex set of $G(\boldsymbol{c})$ is non-empty since it always contains $O$. Assume $(i, j)$ is a maximal vertex, i.e., there is no other vertex $(i', j') \neq (i, j)$ in $G(\boldsymbol{c})$ such that $i' \geqslant i$ and $j' \geqslant j$. This implies that $a_{(i)} \geqslant u_{i,j}$ and $b_{(j)} \geqslant v_{i,j}$. Therefore $a_{(i)} \geqslant u_{i,k}$ for all $0 \leqslant k \leqslant j$, and $b_{(j)} \geqslant v_{l,j}$ for all $0 \leqslant l \leqslant j$. By the definition of $G(\boldsymbol{c})$, none of the $E$-edges connecting $(i, k)$ to $(i + 1, k)$, or the $N$-edges connecting $(l, j)$ to $(l, j + 1)$ is in $G(\boldsymbol{c})$. Thus any vertex of $G(\boldsymbol{c})$, being connected to $O$, must be less than $(i, j)$ under $\preceq$. □

Denote by $v(\boldsymbol{c})$ the unique maximal vertex of $G(\boldsymbol{c})$ under $\preceq$. Partition the set $\mathcal{S}(m, n)$ by $v(\boldsymbol{c})$. i.e., let $K_{m,n}(i, j) \subseteq \mathcal{S}(m, n)$ be the set

$$K_{m,n}(i, j) = \{\boldsymbol{c} = (\boldsymbol{a}, \boldsymbol{b}) \in \mathcal{S}(m, n) : \ v(\boldsymbol{c}) = (i, j)\}.$$

Then $\mathcal{S}(m,n)$ is the disjoint union of $K_{m,n}(i,j)$ for $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$. Comparing the definitions, one notes that the set $K_{m,n}(m,n)$ is exactly the union

$$\bigcup_{P:O\to A} \mathcal{S}_{m,n}(P;\boldsymbol{U}),$$

where $P$ ranges over all lattice paths from $O$ to $A$ using $N$- and $E$-steps only.

Let $k_{m,n}(i,j) = |K_{m,n}(i,j)|$. By convention let $k_{0,0}(0,0) = 1$. We shall prove $k_{m,n}(m,n) = g_{m,n}((x,y); Z_{m,n})$ by showing that they both satisfy the linear recurrence (3.6).

Consider the set $K_{m,n}(i,j)$. A pair $\boldsymbol{c} = (\boldsymbol{a}, \boldsymbol{b}) \in K_{m,n}(i,j)$ means there exists a lattice path $P'$ from $(0,0)$ to $(i,j)$ such that

1. the first $i$ order statistics of $\boldsymbol{a}$ and the first $j$ order statistics of $\boldsymbol{b}$ are bounded by $P'$ with respect to $\boldsymbol{U}$. Let $\boldsymbol{a}'$ be the subsequence of $\boldsymbol{a}$ consisting of the lowest $i$ terms, and $\boldsymbol{b}'$ be the subsequence of $\boldsymbol{b}$ consisting of the lowest $j$ terms. Then the pair $(\boldsymbol{a}', \boldsymbol{b}')$ is in $K_{i,j}(i,j)$. Such pairs are counted by $k_{i,j}(i,j)$.
2. For the sequence $\boldsymbol{a}$, the largest $m-i$ terms all take values in the interval $[u_{i,j}, x)$.
3. For the sequence $\boldsymbol{b}$, the largest $n-j$ terms all take values in the interval $[v_{i,j}, y)$.

Hence

$$k_{m,n}(i,j) = \binom{m}{i}\binom{n}{j} k_{i,j}(i,j)(x-u_{i,j})^{m-i}(y-v_{i,j})^{n-j} = \binom{m}{i}\binom{n}{j} x_{i,j}^{m-i} y_{i,j}^{n-j} k_{i,j}(i,j).$$

Thus

$$x^m y^n = |\mathcal{S}(m,n)| = \sum_{i=0}^{m}\sum_{j=0}^{n} k_{m,n}(i,j) = \sum_{i=0}^{m}\sum_{j=0}^{n} \binom{m}{i}\binom{n}{j} x_{i,j}^{m-i} y_{i,j}^{n-j} k_{i,j}(i,j).$$

Comparing the initial values $k_{0,0}(0,0) = 1 = g_{0,0}((x,y), Z)$, we obtain Theorem 6.1. □

Theorem 6.1 suggests a way to generalize the notion of parking function to higher-dimensions. Let us state the two-dimensional definition.

**Definition 3.** Given a set of nodes $\boldsymbol{U} = \{(u_{i,j}, v_{i,j}) \in \mathbb{N}^2 : i,j \in \mathbb{N}\}$ with the properties that $u_{i,j} \leqslant u_{i',j'}$ and $v_{i,j} \leqslant v_{i;,j'}$ whenever $(i,j) \preceq (i',j')$. A pair $(\boldsymbol{a}, \boldsymbol{b})$ of sequences of length $(m,n)$ is a 2-dimensional $\boldsymbol{U}$-parking function if and only if its order statistics are bounded by some lattice paths from $(0,0)$ to $(m,n)$ with respect to $\boldsymbol{U}$.

Similarly one can define $k$-dimensional $\boldsymbol{U}$-parking functions with respect to a set of nodes $\boldsymbol{U} = \{z_{i_1,\dots,i_k} \in \mathbb{N}^k : i_j \in \mathbb{N}\}$ as $k$-tuples of sequences of length $(n_1,\dots,n_k)$ whose order statistics are bounded by some lattice paths from the origin to $(n_1,\dots,n_k)$ with respect to $\boldsymbol{U}$. To justify this definition, note that in the case of $k=1$, there is a unique lattice path along the $x$-axis from 0 to $n$, hence the integer sequences defined are exactly those whose order statistics are bounded by the given set of nodes. It agrees with the existing notion of $\boldsymbol{u}$-parking functions.

Let $\mathcal{PK}_{m,n}^{(2)}(\boldsymbol{U})$ be the subset of $\mathcal{S}(m,n)$ which consists of all 2-dimensional $\boldsymbol{U}$-parking functions, and $PK_{m,n}^{(2)}(\boldsymbol{U})$ the size of $\mathcal{PK}_{m,n}^{(2)}(\boldsymbol{U})$. Then Theorem 6.1 can be restated as

$$PK_{m,n}(\boldsymbol{U}) = g_{m,n}((x,y);Z) = g_{m,n}((x,y);(x,y)-\boldsymbol{U}).$$

Using the shift invariance formula and the homogeneity of bivariate Gončarov polynomials, we obtain the following theorem.

**Theorem 6.5.**

$$PK_{m,n}^{(2)}(\boldsymbol{U}) = g_{m,n}((x,y);(x,y)-\boldsymbol{U}) = g_{m,n}((0,0);-\boldsymbol{U}) = (-1)^{m+n} g_{m,n}((0,0);\boldsymbol{U}). \tag{6.3}$$

By Theorem 6.5 any result on bivariate Gončarov polynomials yields automatically a result of 2-dimensional parking functions. For example, again using the homogeneity of Gončarov polynomials we have

**Corollary 6.6.**

$$PK^{(2)}_{m,n}(\{(au_{i,j}, bv_{i,j}) : i, j \in \mathbb{N}\}) = a^m b^n PK^{(2)}_{m,n}(\{(u_{ij}, v_{ij}) : i, j \in \mathbb{N}\}).$$

**Remark.** Recall an equivalent definition for $\boldsymbol{u}$-parking functions: a sequence $(a_1, \ldots, a_n)$ is a $\boldsymbol{u}$-parking function if and only if there is a permutation $\sigma$ of length $n$ such that $a_{\sigma(i)} < u_i$. Using this definition we get an interesting relation between 2-dimensional parking functions and the one-dimensional ones. Assume that $u_{i,j} = v_{i,j} = \alpha_{i+j}$ for some sequence $\{\alpha_i\}$. By definition, a pair $(\boldsymbol{a}, \boldsymbol{b})$ of length $(m, n)$ is a 2-dimensional $\boldsymbol{U}$-parking function if and only if there is a rearrangement of the terms $a_i$ and $b_j$ such that it is term-wise bounded by $\alpha_0, \alpha_1, \ldots, \alpha_{m+n-1}$, i.e.,

$$PK^{(2)}_{m,n}(\{(u_{i,j}, v_{i,j}) : u_{i,j} = v_{i,j} = \alpha_{i+j}\}) = PK_{m+n}(\alpha_0, \ldots, a_{m+n-1}).$$

In particular, when $\alpha_k = 1 + k$, the number of 2-dimensional $\boldsymbol{U}$-parking functions is given by a Cayley number

$$PK^{(2)}_{m,n}(\{(u_{i,j}, v_{i,j}) :\ u_{i,j} = v_{i,j} = 1 + i + j\}) = PK_{m+n}(1, 2, \ldots, m + n) = (1 + m + n)^{m+n-1}.$$

When $\alpha_k = a + bk$, we have the Abel evaluation

$$\begin{aligned} PK^{(2)}_{m,n}(\{(u_{i,j}, v_{i,j}) : u_{i,j} = v_{i,j} = a + b(i + j)\}) &= PK_{m+n}(a, a + b, a + 2b, \ldots, a + (m + n - 1)b) \\ &= a(a + (m + n)b)^{m+n-1}. \end{aligned}$$

In general, when $u_{i,j} = a_1 + b_1 i + c_1 j$ and $v_{i,j} = a_2 + b_2 i + c_2 j$ we obtain a 2-dimensional analog of Abel polynomials. Combinatorial identities associated with such polynomials will be discussed in an upcoming paper [6].

As a conclusion, we present an explicit formula for the sum-enumerator of 2-dimensional $\boldsymbol{U}$-parking functions, which may be viewed as a $q$-analog of $PK^{(2)}_{m,n}(\boldsymbol{U})$. More precisely, for any pair of sequences $\boldsymbol{c} = (\boldsymbol{a}, \boldsymbol{b}) \in \mathcal{S}(m, n)$, let $\mathrm{sum}(\boldsymbol{c}) = (p^{\sum_{i=0}^{m-1} a_i}) \cdot (q^{\sum_{j=0}^{n-1} b_j})$. Then

$$[x]_p^m [y]_q^n = \sum_{\boldsymbol{c} \in \mathcal{S}(m,n)} \mathrm{sum}(\boldsymbol{c}) = \sum_{i=0}^{m} \sum_{j=0}^{n} \Bigg( \sum_{\boldsymbol{c} \in K_{m,n}(i,j)} \mathrm{sum}(\boldsymbol{c}) \Bigg), \tag{6.4}$$

where $[x]_p$ is the $p$-integer defined by $[x]_p = 1 + p + \cdots + p^{x-1} = \frac{1-p^x}{1-p}$. Similar definition holds for $[y]_q$.

Using the decomposition described in the proof of Theorem 6.1 and analyzing the sum enumerator of the set $K_{m,n}(i, j)$, we obtain

$$\sum_{\boldsymbol{c} \in K_{m,n}(i,j)} \mathrm{sum}(\boldsymbol{c}) = \binom{m}{i} \binom{n}{j} ([x]_p - [u_{i,j}]_p)^{m-i} ([y]_q - [v_{i,j}]_q)^{n-j} \sum_{\boldsymbol{c} \in K_{i,j}(i,j)} \mathrm{sum}(\boldsymbol{c}). \tag{6.5}$$

Notice that

$$[x]_p - [u_{i,j}]_p = \frac{p^{u_{i,j}} - p^x}{1 - p}, \quad [y]_q - [v_{i,j}]_q = \frac{q^{v_{i,j}} - q^y}{1 - q}.$$

Therefore

$$[x]_p^m [y]_q^n = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \left( \frac{p^{u_{i,j}} - p^x}{1 - p} \right)^{m-i} \left( \frac{q^{v_{i,j}} - q^y}{1 - q} \right)^{n-j} \sum_{\boldsymbol{c} \in K_{i,j}(i,j)} \mathrm{sum}(\boldsymbol{c}). \tag{6.6}$$

Comparing Equations (6.6) and (3.6), and using the shift invariance formula (3.13) and the homogeneity, we have

$$\begin{aligned} \sum_{\boldsymbol{c} \in K_{m,n}(m,n)} \mathrm{sum}(\boldsymbol{c}) &= g_{m,n} \left( \left( \frac{1}{1-p}, \frac{1}{1-q} \right); \left\{ \left( \frac{p^{u_{i,j}}}{1-p}, \frac{q^{v_{i,j}}}{1-q} \right) : i, j \in \mathbb{N} \right\} \right) \\ &= \frac{1}{(1-p)^m (1-q)^n} g_{m,n} \left( (1, 1); Z(\boldsymbol{U})_{[p,q]} \right), \end{aligned} \tag{6.7}$$

where $Z(\boldsymbol{U})_{[p,q]} = \{z_{i,j}[p, q] = (p^{u_{i,j}}, q^{v_{i,j}}) : i, j \in \mathbb{N}\}$. This is summarized in the following theorem, which generalizes the formula for the sum enumerators of parking functions [10, Theorem 6.2].

**Theorem 6.7.**

$$\sum_{\boldsymbol{c}\in\mathcal{PK}^{(2)}_{m,n}(\boldsymbol{U})} p^{\sum_{i=0}^{m-1} a_i}\cdot q^{\sum_{j=0}^{n-1} b_j} = \frac{1}{(1-p)^m(1-q)^n} g_{m,n}((1,1); Z(\boldsymbol{U})_{[p,q]}).$$

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